NEIGHBORHOODS IN STRATIFIED SPACES WITH TWO STRATA

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Abstract. We develop a theory of tubular neighborhoods for the lower strata in manifold stratified spaces with two strata. In these topologically stratified spaces, manifold approximate fibrations and teardrops play the role that fibre bundles and mapping cylinders play in smoothly stratified spaces. Applications include the classification of neighborhood germs, the construction of exotic stratifications, a multiparameter isotopy extension theorem and an h–cobordism extension theorem.

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1. Introduction

The question that motivates this paper is a basic one: suppose that one has a locally flat topological submanifold of a manifold, what kind of geometric structure describes the neighborhood?

For smooth manifolds the entirely satisfactory answer is given by the tubular neighborhood theorem which identifies neighborhood germs with vector bundles.

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In the piecewise linear category, one has the theory of block bundles [51]. For the topological category, the situation is much messier: essentially one can classify the neighborhoods without really describing them (see [52]).

The answer that we give is in terms of a variant of the notion of a fiber bundle, the manifold approximate fibration (MAF). While fiber bundles are maps with identifications of the inverse images of points, MAFs are essentially maps with identifications of the inverse images of open balls. At the level of definitions, they are to fiber bundles what cell-like maps are to homeomorphisms. However, unlike the cell-like case, they cannot always be approximated by bundles (or even block bundles) and represent a genuinely more general notion. Happily, though, one has a good control of the theory of MAFs, see [29], [30].

A special case of our theorem asserts that the (space of) $d+n$ dimensional locally flat germ neighborhoods of an n–manifold M^n are (is homotopy equivalent to the space of) MAFs mapping to $M \times \mathbb{R}$, with the inverse images of small balls in $M \times \mathbb{R}$ homeomorphic to $S^{d-1} \times \mathbb{R}^{n+1}$. One should think of a MAF mapping to $M \times \mathbb{R}$ as having as domain a deleted neighborhood of M and as consisting of two pieces: the first is the projection of generalized tubular neighborhood bundle, and the second is the radial direction, e.g. something like distance from the submanifold. We call this structure a 'teardrop neighborhood.'

Actually, though, our paper is written in more generality. It gives an analysis of neighborhoods of the singular stratum of a stratified space as in [48] which has only two strata. This means that our results apply, for instance to quotients of semifree group actions, and leads to new results for these.

The description of germ neighborhoods is good enough to recover and reprove Quinn's isotopy and homogeneity theorems, and go rather further: we obtain multiparameter isotopy extension theorems, which lead to local contractibility of homeomorphism groups for such spaces.

Another important application is to complete (in the two stratum case) the h–cobordism theorem given in [48]. That paper provides an invariant whose vanishing is necessary and sufficient for a stratified h–cobordism to be a product. We give the realization: any element in the appropriate Whitehead group can be realized by a stratified h–cobordism.

The picture we give of stratified spaces, when combined with the analysis of MAFs in [29] and the stable homeomorphism groups in [63], is more than fine enough to be used to give an independent proof of the two stratum case of the stratified classification results in [62]. However, the current approach is more directly geometric, which has at least two important advantages. The first is that the analysis is done here unstably: i.e. without first crossing with Euclidean spaces and then removing them. We apply this to give examples of stratified spaces (even topological locally linear orbifolds) where no amount of Euclidean stabilization allows one to construct a block bundle neighborhood of the lower stratum.

The other main advantage is that of canonicity, which is important for the multiparameter results discussed above, and also plays a key role in relating the splitting results for spaces of MAFs over Hadamard manifolds proven in [32], and the Novikov rigidity results proven by Ferry and Weinberger (see [16], [17]) for stratified spaces with nonpositively curved strata. These seemingly different results are essentially equivalent after taking a loop space.

Finally, these results form the bottom of an induction that leads to extensions of all of the theorems and applications mentioned above to general stratified spaces

with an arbitrary number of strata (see [26], [27]).

2. Definitions and the main results

Quinn [48] has proposed a setting for the study of those spaces admitting purely topological stratifications as distinct from the smooth stratifications of Whitney [65], Thom [58], Mather [40] and others (cf. [18]). In this paper we consider spaces X containing a manifold B such that the pair (X, B) is a manifold homotopically stratified set in the sense of Quinn. We call X a manifold stratified space with two strata. Roughly, this means that $X \setminus B$ is a manifold, B satisfies a tameness condition in X, and there is a good homotopy model for a normal fibration of B in X.

We begin by recalling the definitions relevant to the manifold stratified spaces. Most of these concepts can be found in Quinn [48] and Weinberger [62], but our terminology is not consistent with either source. Moreover, since we are only dealing with stratified spaces with two strata, our definitions are specialized to that case.

Let (X, A) be a pair of spaces so that $A \subseteq X$. Then X is said to have two *strata*: the lower (or bottom) stratum A and the top stratum $X \setminus A$. If (Y, B) is another pair, then a map $f : (X, A) \to (Y, B)$ is said to be *strict*, or *stratum-preserving*, if $f(X \setminus A) \subseteq Y \setminus B$ and $f(A) \subseteq B$. The subspace A of X is said to be *forward tame* if there exists a neighborhood N of A in X and a strict map $H : (N \times I,\ A \times I \cup N \times \{0\}) \to (X, A)$ such that $H(x, t) = x$ for all $(x, t) \in A \times I$ and $H(x, 1) = x$ for all $x \in N$. In this case, H is called a *nearly strict deformation* of N into A.

Let $\mathrm{Map_s}((X, A), (Y, B))$ denote the space of strict maps with the compact-open topology. The *homotopy link* of A in X is

$$\mathrm{holink}(X, A) = \mathrm{Map_s}(([0, 1], \{0\}),\ (X, A)).$$

Evaluation at 0 defines a map $q : \mathrm{holink}(X, A) \to A$ which should be thought of as a model for a normal fibration of A in X. A point inverse $q^{-1}(x)$ is the *local homotopy link* (or *local holink*) at $x \in A$. In the case that X is an n-manifold and A is a locally flat submanifold of dimension i, then Fadell proved that $q : \mathrm{holink}(X, A) \to A$ is a fibration with homotopy fibre S^{n-i-1} and used the homotopy link as a substitute in the topological category for tubular neighborhoods in the differential category (see [12], [44], [24], [25], [28, App. B].)

The pair (X, A) is said to be a *homotopically stratified pair* if A is forward tame in X and if $q : \mathrm{holink}(X, A) \to A$ is a fibration. If in addition, the fibre of $q : \mathrm{holink}(X, A) \to A$ is finitely dominated, then (X, A) is said to be *homotopically stratified with finitely dominated local holinks*. (When we say that the fibre of q is finitely dominated and A is not path connected, we mean that each fibre of q is finitely dominated.) If the strata A and $X \setminus A$ are manifolds (without boundary), X is a locally compact separable metric space, and (X, A) is homotopically stratified with finitely dominated local holinks, then (X, A) is a *manifold stratified pair*.

We now define the set of equivalence classes of neighborhoods which is the main object of study in this paper. Let B be an i-manifold (without boundary) and let $n \geq 0$ be a fixed integer. A *germ of a stratified neighborhood* of B is an equivalence class represented by a manifold stratified pair (X, B) with $\dim(X \setminus B) = n$. Two such pairs (X, B) and (Y, B) are *germ equivalent* provided that there exist open neighborhoods U and V of B in X and Y, respectively, and a homeomorphism h :

$U \to V$ such that $h|B = \mathrm{id}_B$. In this paper we will classify stratified neighborhoods of B up to germ equivalence (provided $n \geq 5$). The basic construction which makes this possible is now described.

Let $p : X \to Y \times \mathbb{R}$ be a map. The *teardrop* of p, denoted $X \cup_p Y$, is the space with underlying set the disjoint union $X \amalg Y$ and natural topology defined in §3 below. We are interestested in those maps p with the property that $(X \cup_p Y, Y)$ is a manifold stratified or homotopically stratified pair.

Recall that an *approximate fibration* is a map with the approximate homotopy lifting property (see Definition 4.5) and that a map $p : X \to Y$ is a *manifold approximate fibration* if p is an approximate fibration, p is proper, and X and Y are manifolds (without boundary) (see e.g. [29]). Two maps $p : X \to Y$ and $p' : X' \to Y$ are *controlled homeomorphic* if there is a homeomorphism $h : \mathrm{cyl}(p) \to \mathrm{cyl}(p')$ between mapping cylinders such that $h|Y = \mathrm{id}_Y$ which is *level* in the sense that h commutes with the natural projections to $[0,1]$. In [29] manifold approximate fibrations over Y with total space of dimension greater than four are classified up to controlled homeomorphism.

The main results can now be stated. Let $n \geq 5$ be a fixed integer and let B be a closed manifold. In the general setting of manifold stratified pairs (X, B), neighborhoods of B in X need not have nice geometric structure. For example, B need not be locally conelike in X and B may even fail to have mapping cylinder neighborhoods (locally or globally). However, the first theorem says that the lower stratum in a manifold stratified pair has a neighborhood which is the teardrop of a manifold approximate fibration. The second theorem is just a more complete statement.

Theorem 2.1 (Teardrop Neighborhood Existence). *Let (X, B) be a pair such that $X \setminus B$ is a manifold of dimension n. Then (X, B) is a manifold stratified pair if and only if B has a neighborhood in X which is the teardrop of a manifold approximate fibration.*

There are two equivalent ways to understand what it means for B to have a neighborhood in X which is the teardrop of a manifold approximate fibration as in Theorem 2.1:

(i) There exist a neighborhood U of B in X and a manifold approximate fibration $p : V \to B \times \mathbb{R}$ such that (U, B) is homeomorphic to $(V \cup_p B, B)$ rel B.

(ii) There exists an open neighborhood U of B in X and a proper map $f : U \to B \times (-\infty, +\infty]$ such that $f^{-1}(B \times \{+\infty\}) = B$, $f| : B \to B \times \{+\infty\}$ is the identity, and $f| : U \setminus B \to B \times \mathbb{R}$ is a manifold approximate fibration.

That these are equivalent follows from the material in §3 (see especially Proposition 3.7). Theorem 2.1 follows directly from the following theorem.

Theorem 2.2 (Neighborhood Germ Classification). *The teardrop construction defines a bijection from the set of controlled homeomorphism classes of manifold approximate fibrations over $B \times \mathbb{R}$ (with total space of dimension n) to the set of germs of stratified neighborhoods of B (with top stratum of dimension n).*

In fact, Theorem 2.2 is just the consequence at the π_0 level of a more general Higher Classification Theorem which asserts that two simplicial sets are homotopy equivalent (Theorem 2.3 below). However, a proof of Teardrop Neighborhood

Existence (Theorem 2.1) is offered in §7 which avoids some of the parametric considerations needed for Theorem 2.3. Before we can define the simplicial sets appearing in Theorem 2.3 we need sliced versions of some of the definitions.

Let Δ be a space which will play the role of a parameter space. Let $(X, A\times\Delta)$ be a pair of spaces and let $\pi : X \to \Delta$ be a map such that $\pi| : A \times \Delta \to \Delta$ is the projection. Then $A\times\Delta$ is said to be *sliced forward tame* in X (with respect to π) if there exists a neighborhood N of $A\times\Delta$ in X and a nearly strict deformation H of N into $A\times\Delta$ such that H is fibre preserving over Δ (i.e., $\pi H_t = \pi$ for all $t \in I$). The *sliced homotopy link* of $A\times\Delta$ in X (with respect to π) is $\operatorname{holink}_\pi(X, A\times\Delta) = \{\omega \in \operatorname{Map_s}(([0,1],\{0\}),(X, A\times\Delta)) \mid \pi\omega(t) = \pi\omega(0) \text{ for all } t \in I\}$. Note that evaluation at 0 still gives a map $q : \operatorname{holink}_\pi(X, A\times\Delta) \to A\times\Delta$.

Let $n \geq 0$ be a fixed integer and let B be a manifold (without boundary). In §5 the simplicial set $\mathrm{SN}^n(B)$ of stratified neighborhoods of B is defined. Roughly, its k-simplices are k-parameter families of manifold stratified spaces containing $B\times\Delta^k$ as the lower stratum using the notions of sliced forward tameness and the sliced homotopy link. On the other hand, the simplicial set $\mathrm{MAF}^n(B\times\mathbb{R})$ of manifold approximate fibrations over $B\times\mathbb{R}$ was defined in [29] (see also §5). This set has k-simplices consisting of k-parameter families of manifold approximate fibrations over $B\times\mathbb{R}$.

Note that if $p : M \to B\times\mathbb{R}\times\Delta^k$ is a map, then the teardrop construction yields a pair $(M\cup_p B\times\Delta^k, B\times\Delta^k)$. Define $\Psi(p) = (M\cup_p B\times\Delta^k, B\times\Delta^k)$. The following result is the simplicial set version of Theorem 2.2.

Theorem 2.3 (Higher Classification). *If B is a closed manifold and $n \geq 5$, then the teardrop construction defines a homotopy equivalence $\Psi : \mathrm{MAF}^n(B\times\mathbb{R}) \to \mathrm{SN}^n(B)$.*

To see why Theorem 2.2 follows from Theorem 2.3, recall that $\pi_0 \mathrm{MAF}^n(B\times\mathbb{R})$ is the set of controlled homeomorphism classes of manifold approximate fibrations over $B\times\mathbb{R}$ (see [29]). And it is not difficult to see that $\pi_0 \mathrm{SN}^n(B)$ is the set of germs of stratified neighborhoods of B (see Corollary 5.6).

Fibre bundles have well-defined fibres up to homeomorphism. Analogously, manifold approximate fibrations have well-defined fibre germs up to controlled homeomorphism (see [29]). Recall that if $p : M \to B$ is a manifold approximate fibration with B connected, $\dim B = i$ and $\dim M = n \geq 5$, then the *fibre germ* of p is the manifold approximate fibration $q = p| : V = p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i$ where $\mathbb{R}^i \hookrightarrow B$ is an open embedding (which is orientation preserving if B is oriented). The theorems above involve manifold approximate fibrations $p : M \to B\times\mathbb{R}$ and these have fibre germs of the form $q : V \to \mathbb{R}^{i+1}$. The teardrop construction yields a manifold stratified pair $(V\cup_q \mathbb{R}^i, \mathbb{R}^i) \subseteq (M\cup_p B, B)$. The local holink of B in $M\cup_p B$ is homotopy equivalent to V. For locally conelike stratified pairs (X, B) (see [55]) a neighborhood of B in X is given by the teardrop of a manifold approximate fibration $p : M \to B\times\mathbb{R}$ with *trivial fibre germ*; that is, the projection $F\times\mathbb{R}^{i+1} \to \mathbb{R}^{i+1}$ for some closed manifold F.

Let $\mathrm{MAF}(B\times\mathbb{R})_q$ be the simplicial subset of $\mathrm{MAF}^n(B\times\mathbb{R})$ consisting of manifold approximate fibrations with fibre germ $q : V \to \mathbb{R}^{i+1}$. For trivial fibre germ, we write this simplicial set as $\mathrm{MAF}(B\times\mathbb{R})_{F\times\mathbb{R}^{i+1}}$. According to [29], [30], $\mathrm{MAF}(B\times\mathbb{R})_q$ is homotopy equivalent to a simplicial set of lifts of $B \to \mathrm{BTOP}_{i+1}$ up to $\mathrm{BTOP}^{\mathrm{level}}(q)$ where $B \to \mathrm{BTOP}_{i+1}$ is the composition of the classifying map $B \to \mathrm{BTOP}_i$ for the tangent bundle of B with the map $\mathrm{BTOP}_i \to \mathrm{BTOP}_{i+1}$ induced by

euclidean stabilization. The fibre of $\mathrm{BTOP}^{\mathrm{level}}(q) \to \mathrm{BTOP}_{i+1}$ is $\mathrm{BTOP}^c(q)$, the classifying space of controlled homeomorphisms on $q : V \to \mathbb{R}^{i+1}$. According to [31] $\mathrm{BTOP}^c(q) \simeq \mathrm{BTOP}^b(q)$, the classifying space of bounded homeomorphisms. In the case of trivial fibre germ $F \times \mathbb{R}^{i+1} \to \mathbb{R}^{i+1}$, this is written as $\mathrm{BTOP}^b(F \times \mathbb{R}^{i+1})$. For relevant information about the homotopy type of $\mathrm{BTOP}^b(F \times \mathbb{R}^{i+1})$ see [63]. For example, if $B \times \mathbb{R}$ is parallelizable, then

$$\mathrm{MAF}(B \times \mathbb{R})_{F\times\mathbb{R}^{i+1}} \simeq \mathrm{Map}(B, \mathrm{BTOP}^b(F \times \mathbb{R}^{i+1}))$$

and this classifies neighborhood germs in the locally conelike case.

These classification results together with [63] can be used to give an alternative proof of Weinberger's surgery theoretic stable classification theorem [62] in the case of two strata. In fact, this alternative proof is outlined in [62, 10.3.A] and discussed here in Remark 8.17(i).

In §8 the classification theorem for manifold approximate fibrations is combined with the classification of neighborhood germs to construct examples of manifold stratified pairs in which the lower strata do not have a neighborhood given by the mapping cylinder of a fibre bundle, or even a block bundle. Moreover, the examples do not improve in this regard under euclidean stabilization. These examples are locally conelike and the lower strata do have neighborhoods which are mapping cylinders of manifold approximate fibrations.

In addition, Theorem 2.2 provides the link between the results on approximate fibrations proven in [32] and the tangentiality results of [16], [17].

Teardrop neighborhoods can also be used in conjunction with the geometric theory of manifold approximate fibrations [22], [24] to study the geometric topology of manifold stratified pairs. We include two examples here, both of which involve extending a structure on the lower stratum to a neighborhood of the stratum. This is a very important use of manifold approximate fibrations which is similar to the way fibre bundles are used in inductive proofs for smoothly stratified spaces. The following isotopy extension theorem is established in §9.

Corollary 2.4 (Parametrized Isotopy Extension). *If (X, B) is a manifold stratified pair, $\dim X \geq 5$, B is a closed manifold and $h : B \times \Delta^k \to B \times \Delta^k$ is a k-parameter isotopy (i.e., h is a homeomorphism, fibre preserving over Δ^k, and $h|B\times\{0\} = \mathrm{id}_{B\times\{0\}}$), then there exists a k-parameter isotopy $\tilde{h} : X\times\Delta^k \to X\times\Delta^k$ extending h such that $\tilde{h}$ is the identity on the complement of an arbitrarily small neighborhood of B.*

In the case that B is a locally flat submanifold of X, this theorem is due to Edwards and Kirby [11]. For locally conelike stratified spaces with an arbitrary number of strata, it is due to Siebenmann [55]. Finally, Quinn [48] proved this theorem for manifold stratified spaces in general (with an arbitrary number of strata), but only in the case $k = 1$.

Also in §9 we prove an h–cobordism extension theorem which can be used to prove a realization theorem for stratified Whitehead torsions (see Remark 9.4(i)).

A *fibre preserving map (f.p.)* is a map which preserves the fibres of maps to a given parameter space. The parameter space will usually be a k-simplex or an arbitrary space denoted K. Specifically, if $\rho : X \to K$ and $\sigma : Y \to K$ are maps, then a map $f : X \to Y$ is f.p. (or f.p. over K) if $\sigma f = \rho$.

There is a notion of reverse tameness which, in the presence of forward tameness, is often equivalent to the finite domination of local holinks condition discussed above. See [48, 2.15] and [28, 9.15, 9.17, 9.18] paying special attention to the point-set topological conditions appearing in [28]. Moreover, when strata are manifolds, the notions of forward tameness and reverse tameness are often equivalent (by Poincaré duality). See [48, 2.14] and [28, 10.13, 10.14] paying special attention to the π_1 conditions appearing in [28].

Hughes and Ranicki's book [28] contains many of the the results of this paper in the special case of stratified pairs with lower stratum a single point. The reader is advised to consult that work for background, examples and historical remarks. The paper [27] contains generalizations to manifold stratified spaces with more than two strata. The proofs in [27] are often by induction on the number of strata and rely on the present paper for the beginning of the induction. More applications to the geometric topology of manifold stratified spaces are contained in [27]. See also [26].

3. The topology of the teardrop

Let $p : X \to Y \times \mathbb{R}$ be a map. The *teardrop* of p, denoted by $X \cup_p Y$, is defined to be the space with underlying set the disjoint union $X \amalg Y$ and topology given as follows. First, let $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ be defined by

$$c(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty), & \text{if } x \in Y. \end{cases}$$

Then the topology on $X \cup_p Y$ is the minimal topology such that

(i) $X \subseteq X \cup_p Y$ is an open embedding, and
(ii) c is continuous.

The mapping c is called the *collapse* mapping for $X \cup_p Y$.

Note that a basis for this topology is given by

$$\{c^{-1}(U) \mid U \text{is open in } \ Y \times (-\infty, +\infty]\} \cup \{U \mid U \text{ is open in } X\}.$$

There are two minor variations on this construction which we will use. The first occurs when U is an open subset of X and p is only defined on U, $p : U \to Y \times \mathbb{R}$. Then we let $X \cup_p Y = X \cup (U \cup_p Y)$. The second variation occurs when the range of p is restricted, usually to $Y \times [0, +\infty)$. We can still form $X \cup_p Y$ and the collapse map $c : X \cup_p Y \to Y \times [0, +\infty]$.

Special cases and variations of the teardrop construction have appeared frequently in the literature and we now discuss some examples.

3.1. Mapping cylinders. If $q : X \to Y$ is a map, let $p : X \times (0, 1) \to Y \times (0, 1)$ denote $q \times \mathrm{id}$. Then we define the *open mapping cylinder* of q to be the teardrop

$$\overset{\circ}{\mathrm{cyl}}(q) = (X \times (0, 1)) \cup_p Y,$$

where we replace $\mathbb{R}$ with $(0, 1)$. The *mapping cylinder* is

$$\mathrm{cyl}(q) = (X \times [0, 1)) \cup_p Y.$$

Note that this is not the usual quotient topology on the mapping cylinder (except in special cases), but is more useful geometrically (see [6], [45], [48]). The *open cone*

$\overset{\circ}{c}(X)$ of a space X is just the open mapping cylinder (with the teardrop topology) of the constant map $X \to \{v\}$ with v the vertex of the cone.

It follows from this example that the teardrop $X \cup_p Y$ of a map $p : X \to Y \times [0,1)$ is a mapping cylinder neighborhood of Y if there exist a space Z, a map $q : Z \to Y$, and a homeomorphism $h : Z \times [0,1) \to X$ such that $ph = q \times \mathrm{id}_{[0,1)}$.

3.2. Joins. The join of two spaces $X * Y$ can be viewed as a teardrop as follows. Let $p : X \times (0,1) \times Y \to Y \times (0,1)$ be defined by $p(x,t,y) = (y,t)$. Identify $X \times (0,1)$ with $\overset{\circ}{c}(X) \setminus \{v\}$. Then $X * Y = (\overset{\circ}{c}(X) \times Y) \cup_p Y$. Again, this is not the quotient topology, but it is a topology which is often used.

3.3. Hadamard's teardrop. Let H be an Hadamard manifold of dimension n (i.e., H is a complete, simply connected Riemannian manifold of nonpositive curvature) with distance function d induced by the metric. Fix a point $x_0 \in H$ and let S denote the unit tangent sphere of H at x_0. For each $x \neq x_0$ in H, let $\gamma_x : [0, +\infty) \to H$ be the unique unit speed geodesic such that $\gamma_x(0) = x_0$ and $\gamma_x(d(x_0, x)) = x$. Define $p : H \setminus \{x_0\} \to S \times (0, +\infty)$ by

$$p(x) = (\gamma_x'(0), d(x_0, x)).$$

(It follows from standard facts that $\gamma_x'(0)$ depends continuously on x.) It is easy to see that the teardrop $H \cup_p S$ is homeomorphic to the Eberlein-O'Neill compactification $\overline{H} = H \cup H(\infty)$ with the cone topology [10] (in particular, $H \cup_p S$ is an n-cell). To see this, let $f : [0,1] \to [0,+\infty]$ be a homeomorphism, let B be the unit tangent ball of H at x_0 and let $\psi : B \to H \cup_p S$ be defined by

$$\psi(v) = \begin{cases} \exp(f(\|v\| \cdot v) & \text{if } x \notin S \\ v & \text{if } x \in S. \end{cases}$$

Then ψ is a homeomorphism (using the continuity criterion below) and together with [10, Prop. 2.10] can be used to get a homeomorphism with $\overline{H}$.

Another useful construction is as follows. If $q : M \to H$ is a map, then the composition $pq : M \setminus q^{-1}(x_0) \to S \times (0, +\infty)$ yields a teardrop $M \cup_{pq} S$. If q is proper, this amounts to compactifying M by adding the sphere $S \approx H(\infty)$ at infinity. This special case of the teardrop was used in [31] for studying manifold approximate fibrations over H.

Point-set topology. A pleasant feature of the teardrop topology is that it is easy to decide when a function into a teardrop is continuous. In fact, the proof of the following lemma follows immediately from the description of the basis above.

Lemma 3.4 (Continuity Criteria). *Let $f : Z \to X \cup_p Y$ be a function. Then f is continuous if and only if*

(i) $f| : f^{-1}(X) \to X$ *is continuous, and*
(ii) *the composition* $Z \xrightarrow{f} X \cup_p Y \xrightarrow{c} Y \times (-\infty, +\infty]$ *is continuous.* □

If (X, Y) is a pair of spaces, we now address the question of the existence of a map $p : X \setminus Y \to Y \times \mathbb{R}$ such that the identity from X to $(X \setminus Y) \cup_p Y$ is a homeomorphism. If this is the case, then (X, Y) is said to be *the teardrop of p*. The answers are in Corollaries 3.11 and 3.12.

If $f : X \to Y$ is a map and $A \subseteq Y$, then f is said to be a *closed mapping over* A if for each $y \in A$ and closed subset K of X such that $K \cap f^{-1}(y) = \emptyset$, it follows that $y \notin \mathrm{cl}(f(K))$ (the closure of $f(K)$).

Remark 3.5.

(i) $f : X \to Y$ is a closed mapping if and only if f is a closed mapping over Y.
(ii) If $A \subseteq Y$ and $f : X \to Y$ is a closed mapping over A, then f is a closed mapping over any $B \subseteq A$.
(iii) If A is closed in Y and $f : X \to Y$ is a closed mapping over A, then $f| : f^{-1}(A) \to A$ is a closed mapping (but not conversely).

Lemma 3.6. *If $p : X \to Y \times \mathbb{R}$ is a map, then the collapse $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ is a closed mapping over $Y \times \{+\infty\}$.*

Proof. Let $y \in Y$ and let K be a closed subset of $X \cup_p Y$ such that $y \notin K$ (note $y = c^{-1}(y, +\infty)$). Then $y \in U = (X \cup_p Y) \setminus K$ and U is open. By the definition of the teardrop topology, there is an open subset V of $(y, +\infty)$ in $Y \times (-\infty, +\infty]$ such that $y \in c^{-1}(V) \subseteq U$. Then $c(K) \cap V = \emptyset$, so $(y, +\infty) \notin \mathrm{cl}(c(K))$. $\square$

Proposition 3.7. *Let (X, Y) be a pair of spaces for which there is a mapping $f : X \to Y \times (-\infty, +\infty]$ such that $f(y) = (y, +\infty)$ for each $y \in Y$ and $f(X \setminus Y) \subseteq Y \times \mathbb{R}$. Let*

$$p = f| : X \setminus Y \to Y \times \mathbb{R}.$$

Then (X, Y) is the teardrop of p if and only if f is a closed mapping over $Y \times \{+\infty\}$.

Proof. First note that f is the collapse c for the teardrop $(X \setminus Y) \cup_p Y$. It follows that the identity $X \to (X \setminus Y) \cup_p Y$ is always continuous. To prove the proposition, assume that the identity is a homeomorphism. By Lemma 3.6, c is a closed mapping over $Y \times \{+\infty\}$. Since $f = c$, so is f.
Conversely, assume f is a closed mapping over $Y \times \{+\infty\}$. Given an open subset U of X, we will show that U is open in $(X \setminus Y) \cup_p Y$. For this, it suffices to consider $y \in U \cap Y$ and show that U is a neighborhood of y in $(X \setminus Y) \cup_p Y$. To this end let $K = X \setminus U$ and observe that since $f^{-1}(y, +\infty) = y \notin K$, it follows that $(y, +\infty) \notin \mathrm{cl}(f(K))$. Thus, there is an open subset V of $Y \times (-\infty, +\infty]$ such that $(y, +\infty) \in V$ and $V \cap f(K) = \emptyset$. Then $c^{-1}(V)$ is open in $(X \setminus Y) \cup_p Y$ and $y \in c^{-1}(V) \subseteq U$. $\square$

Corollary 3.8. *A pair (X, Y) is a teardrop if and only if there is a map $f : X \to Y \times (-\infty, +\infty]$ which is closed over $Y \times \{+\infty\}$ such that $f(y) = (y, +\infty)$ for each $y \in Y$ and $f(x) \in Y \times \mathbb{R}$ for each $x \in X \setminus Y$.* $\square$

Proposition 3.9. *Let (X, Y) be a pair of spaces such that X is Hausdorff and Y is locally compact. Suppose there exist a proper retraction $r : X \to Y$ and a map $\phi : X \to (-\infty, +\infty]$ such that $\phi^{-1}(+\infty) = Y$. Then $f = r \times \phi : X \to Y \times (-\infty, +\infty]$ is a closed mapping over $Y \times \{+\infty\}$. Consequently, (X, Y) is a teardrop.*

Proof. Let $y \in Y$ and let K be a closed subset of X such that $y \notin K$. We need to show that $(y, +\infty) \notin \mathrm{cl}(f(K))$. To this end, let U be open in X such that $y \in U$ and $U \cap K = \phi$. Choose an open subset V of Y such that $y \in V, \mathrm{cl}(V) \subseteq U \cap Y$, and $\mathrm{cl}(V)$ is compact. Let $K_1 = r^{-1}(\mathrm{cl}(V)) \cap K$ and $K_2 = K \setminus r^{-1}(V)$. Then K_1 is compact and $K = K_1 \cup K_2$. Since $f(K_1)$ is compact and $(y, +\infty) \notin f(K_1)$,

it suffices to show that $(y, +\infty) \notin \mathrm{cl}(f(K_2))$. But $(y, +\infty) \in V \times (-\infty, +\infty]$ and $f(K_2) \cap V \times (-\infty, +\infty] = \emptyset$. That (X, Y) is a teardrop follows form Proposition 3.7. $\square$

Note that such a map ϕ in the hypothesis of Proposition 3.9 would exist whenever X is normal and Y is a closed G_δ-subset.

Theorem 3.10. *Let Y be a closed subset of the metrizable space X. Then (X, Y) is a teardrop if and only if there exists a metric d for X and a retraction $r : X \to Y$ such that whenever $\{x_n\}$ is a sequence in X with $x_n \to \infty$ (i.e., $\{x_n\}$ has no convergent subsequence) and $d(x_n, Y) \to 0$, it follows that $r(x_n) \to \infty$.*

Proof. Suppose first the (X, Y) is the teardrop of $p : X \setminus Y \to Y \times \mathbb{R}$ and let $c : X \to Y \times (-\infty, +\infty]$ be the collapse. Define $\rho : X \to [0, +\infty)$ to be the composition

$$X \xrightarrow{c} Y \times (-\infty, +\infty] \xrightarrow{\text{proj}} (-\infty, +\infty] \xrightarrow{h} [0, +\infty)$$

where h is a homeomorphism. Let D be any metric on X and define d by

$$d(x, x') = D(x, x') + |\rho(x) - \rho(x')|.$$

It is easy to see that d is indeed a metric and yields the same topology on X as D. Define $r : X \to Y$ to be the composition

$$X \xrightarrow{c} Y \times (-\infty, +\infty] \xrightarrow{\text{proj}} Y.$$

To see that r has the desired property, let $\{x_n\}$ be a sequence in X such that $x_n \to \infty$ and $d(x_n, Y) \to 0$. Given $y \in Y$ we will show that there is no subsequence $\{x_{n_k}\}$ with $r(x_{n_k}) \to y$. To this end let

$$K = \bigcup_{n=1}^{\infty} \{x_n\} \setminus \{y\}.$$

Then K is a closed subset of X and $y \notin K$. Since c closed over $Y \times \{+\infty\}$ by Lemma 3.6, it follows that $(y, +\infty) \notin \mathrm{cl}(c(K))$. Thus, if $\{x_{n_k}\}$ is a subsequence, $\{c(x_{n_k})\}$ does not converge to $(y, +\infty)$. Since $d(x_n, Y) \to 0$, $\rho(x_n) \to 0$. This implies $c(x_n) \to Y \times \{+\infty\}$. If $r(x_{n_k}) \to y$, then we would have $c(x_{n_k}) \to (y, +\infty)$, a contradiction.
Conversely, assume r and d are given as above. Define $\phi : X \to (-\infty, +\infty]$ by

$$\phi(x) = \begin{cases} \frac{1}{d(x,Y)}, & \text{if } x \in X \setminus Y \\ +\infty & \text{if } x \in Y. \end{cases}$$

Let $f = r \times \phi : X \to Y \times (-\infty, +\infty]$. By Corollary 3.8, it suffices to show that f is closed over $Y \times \{+\infty\}$. To this end let K be closed in X and $y \in Y \setminus K$. Suppose $(y, +\infty) \in \mathrm{cl}(f(K))$. Then there exists a sequence $\{x_n\}$ in K such that $f(x_n) \to (y, +\infty)$. Then $r(x_n) \to y$ and $\phi(x_n) \to +\infty$. Thus, $d(x_n, Y) \to 0$. If $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, then $x_{n_k} \to y_0 \in Y \cap K$. Then $r(x_{n_k}) \to y_0$ so $y = y_0$, a contradiction since $y \notin K$. Thus, we must have $x_n \to \infty$. So $r(x_n) \to \infty$, again a contradiction. $\square$

Corollary 3.11. *If Y is a compact subset of the metric space X, then (X, Y) is a teardrop if and only if there exists a retraction $r : X \to Y$.* $\square$

Corollary 3.12. *Let Y be a closed subset of the locally compact metric space X. Then (X, Y) is a teardrop if and only if there exists a retraction $r : X \to Y$.*

Proof. If (X, Y) is a teardrop, let r be given by Theorem 3.10. Conversely, if $r : X \to Y$ is a retraction then by Proposition 3.9, it suffices to show that Y has a closed neighborhood N in X such that $r| : N \to Y$ is proper. To this end, for each $y \in Y$, let N_y be a compact neighborhood of y in X and let

$$N = \bigcup\{r^{-1}(N_y \cap Y) \cap N_y \mid y \in Y\}. \quad \square$$

We now observe that there are versions of the preceding results which are valid near Y. To make this precise, let (X, Y) be a pair of spaces. An open neighborhood U of Y in X is said to be a *teardrop neighborhood* if the pair (U, Y) is a teardrop; that is, there is a map

$$p : U \setminus Y \to Y \times \mathbb{R}$$

such that the identity from X to $(X \setminus Y) \cup_p Y$ is a homeomorphism. The following results follow immediately from Corollaries 3.11 and 3.12.

Corollary 3.13. *If Y is a compact subset of the metric space X, then Y has a teardrop neighborhood in X if and only if Y is a neighborhood retract of X.* $\square$

Corollary 3.14. *Let Y be a closed subset of the locally compact metric space X. Then Y has a teardrop neighborhood in X if and only if Y is a neighborhood retract of X.* $\square$

Next, we prove a lemma which will be useful in §4.

Lemma 3.15. *If X and Y are metric spaces and $p : X \to Y \times \mathbb{R}$ is a map, then the teardrop $X \cup_p Y$ is metrizable.*

Proof. Let d_X and d_Y be metrics for X and Y, respectively. Define a function $\rho : (X \amalg Y) \times (X \amalg Y) \to [0, +\infty)$ by

$$\rho(a, b) = \begin{cases} d_X(a, b), & \text{if } a, b \in X \\ d_Y(a, b), & \text{if } a, b \in Y \\ 0, & \text{otherwise.} \end{cases}$$

Define a metric d on $Y \times (-\infty, +\infty]$ by

$$d((y_1, t_1), (y_2, t_2)) = \max\{d_Y(y_1, y_2),\ |e^{-t_1} - e^{-t_2}|\}$$

where $e^{-\infty} = 0$. Note that d generates the standard topology. Define the metric D on $X \cup_p Y$ by

$$D(a, b) = \rho(a, b) + d(c(a), c(b))$$

where $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ is the usual collapse. One checks that D generates the teardrop topology. $\square$

Related constructions. Whyburn appears to be the first to have considered a construction similar to the teardrop (see [66], [67]), and we now explore the relationship between the two constructions. Let X and Y denote (disjoint) Hausdorff spaces and let $f : X \to Y$ be a map. Whyburn defines the *unified space* $X \oplus Y$ to be the topological space whose underlying set is $X \amalg Y$ and with topology $\mathcal{W}$ given by $V \in \mathcal{W}$ if and only if

(i) $V \cap X,\ V \cap Y$ are open in X, Y, respectively, and
(ii) for every compact $K \subseteq V \cap Y,\ f^{-1}(K) \cap (X \setminus V)$ is compact.

Whyburn proved that the function $r : X \oplus Y \to Y$ defined by $r(x) = f(x)$ if $x \in X$, and $r(y) = y$ if $y \in Y$, is a (continuous) proper retraction.

For the next two propositions suppose $p : X \to Y \times [0, +\infty)$ is a map and $f : X \to Y$ is the composition

$$X \xrightarrow{p} Y \times [0, +\infty) \xrightarrow{\text{proj}} Y.$$

Proposition 3.16. *If X and Y are Hausdorff, then the identity from $X \oplus Y$ to $X \cup_p Y$ is continuous if and only if p is proper.*

Proof. Assume first that the identity is continuous, and note that there is a commutative diagram of maps:

$$\begin{array}{ccc} X \oplus Y & \xrightarrow{r} & Y \\ {\scriptstyle \text{id}}\downarrow & & \uparrow{\scriptstyle \text{proj}} \\ X \cup_p Y & \xrightarrow{c} & Y \times [0 + \infty] \end{array}$$

Recall that Whyburn showed r is proper. It follows easily that c is proper. Hence, $p = c| : X \to Y \times [0, +\infty)$ is proper.

Conversely, assume that p is proper. By Lemma 3.4, it suffices to show that the function $q : X \oplus Y \to Y \times [0, +\infty]$ defined by

$$q(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty) & \text{if } x \in Y \end{cases}$$

is continuous. For this we need to show $q^{-1}(U \times (t, +\infty])$ is open where U is open in Y and $t \in [0, +\infty)$. Let $V = q^{-1}(U \times (t, +\infty]) = p^{-1}(U \times (t, +\infty)) \cup U$ and let K be a compact subset of $V \cap Y = U$. Then

$$f^{-1}(K) \cap (X \setminus V) = p^{-1}(K \times [0, +\infty)) \cap (X \setminus V) = p^{-1}(X \times [0, t])$$

which is compact since p is proper. Hence, V is open. $\square$

Proposition 3.17. *If X and Y are Hausdorff, Y is locally compact, and p is proper, then the identity from $X \cup_p Y$ to $X \oplus Y$ is continuous.*

Proof. Let U be open in $X \oplus Y$. To show U is open in $X \cup_p Y$, it suffices to consider $y \in U \cap Y$ and show U is a neighborhood of y in $X \cup_p Y$. Let $y \in V \subseteq \text{cl}(V) \subseteq U \cap Y$ where V is open and $\text{cl}(V)$ is compact. Then $c^{-1}(V \times [0, +\infty])$ is open in $X \cup_p Y$ and $f^{-1}(\text{cl}(V)) \cap (X \setminus U)$ is a compact subset of X. Let

$$W = c^{-1}(V \times [0, +\infty]) \setminus [f^{-1}(\text{cl}(V)) \cap (X \setminus U)].$$

Then $y \in W$, W is open in $X \cup_p Y$ and $W \subseteq U$. $\square$

Corollary 3.18. *If X and Y are Hausdorff and Y is locally compact, then the identity from $X \oplus Y$ to $X \cup_p Y$ is a homeomorphism if and only if p is proper.* $\square$

Many authors ([14], [15], [36], [53]) have used a construction closely related to Whyburn's unified space and we now briefly discuss their construction. Suppose X, Y are disjoint, Hausdorff, spaces, X is locally compact and non-compact, Y is compact, N is a neighborhood of infinity in X, and $f : N \to Y$ is a map. Then Ferry and Pedersen [15] define a space $X \amalg_f Y$ whose underlying set is $X \amalg Y$ and with topology generated by the basis

$$\{V \mid V \text{ is open in } X\} \cup \{(f^{-1}(V) \cap V') \cup V \mid V \text{ is open in } Y \text{ and } V' \text{ is an open neighborhood of infinity in } X\}.$$

It is easy to see that the identity from $X \cup (N \oplus Y)$ to $X \amalg_f Y$ is a homeomorphism. For an alternate treatment of related constructions, one should consult James [34, §8].

Controlled maps. Finally, we use the teardrop topology to clarify the notion of a controlled map given in [29, §12]. For notation, if α is any map we will let $\mathrm{M}(\alpha)$ denote the mapping cylinder of α with the standard quotient topology. On the other hand, $\mathrm{cyl}(\alpha)$ will denote the mapping cylinder with the teardrop topology as in 3.1. Suppose $f_t : X_1 \to X_2$, $0 \le t < 1$, is a family of maps such that the induced map $f : X_1 \times [0, 1) \to X_2$ is continuous. Let $p : X_1 \to Y$ and $q : X_2 \to Y$ be given maps.

Proposition 3.19. *The following are equivalent*:

(i) *f_t is a controlled map from p to q; i.e., $\hat{f} : X_1 \times [0, 1] \to Y$ given by*

$$\hat{f}(x, t) = \begin{cases} q f_t(x), & \text{if } t < 1 \\ p(x), & \text{if } t = 1 \end{cases}$$

is continuous.

(ii) *$f_* : X_1 \times [0, 1] \to \mathrm{cyl}(q)$ given by*

$$f_*(x, t) = \begin{cases} (f_t(x), t), & \text{if } t < 1 \\ p(x), & \text{if } t = 1 \end{cases}$$

is continuous.

(iii) *$\tilde{f} : \mathrm{M}(p) \to \mathrm{cyl}(q)$ given by*

$$\begin{cases} \tilde{f}([x, t]) = (f_t(x), t), & \text{if } t < 1 \\ \tilde{f}([y]) = y, & \text{if } y \in Y \end{cases}$$

is continuous.

Proof. (i) *implies* (ii): Since $\hat{f}$ is continuous, so is $cf_* : X_1 \times [0, 1] \to Y \times [0, 1]$. Lemma 3.4 then implies f_* is continuous.
(ii) *implies* (iii): Let $\pi : (X_1 \times [0, 1]) \amalg Y \to \mathrm{M}(p)$ be the quotient map. Then $\tilde{f}$ is continuous if $\pi\tilde{f}$ is. But $\pi\tilde{f}|X_1 \times [0, 1] = f_*$ and $\pi\tilde{f}|Y$ is the inclusion.
(iii) *implies* (i): Note that $\hat{f}$ is the composition

$$X_1 \times [0, 1] \xrightarrow{\pi} \mathrm{M}(p) \xrightarrow{\tilde{f}} \mathrm{cyl}(q) \xrightarrow{c} Y \times [0, 1] \xrightarrow{\mathrm{proj}} Y. \quad \square$$

4. The teardrop of an approximate fibration

In this section we study the teardrop of an approximate fibration $p : X \to Y \times \mathbb{R}$ and establish two important properties. First, if X and Y are metric spaces, then the teardrop $(X \cup_p Y, Y)$ is a homotopically stratified pair (Theorem 4.7). Second, if p is a manifold approximate fibration, then $(X \cup_p Y, Y)$ is a manifold stratified pair (Corollary 4.11). This second result is part of Theorem 2.1 and does not require the assumption that the dimension be greater than 4. The main technical tool is Theorem 4.2 which characterizes a homotopically stratified pair in terms of a certain lifting property. There are two other useful results. One (Proposition 4.4) shows that the property of being a homotopically stratified pair depends only on a neighborhood of the lower stratum. The other (Proposition 4.8) characterizes (up to fibre homotopy equivalence) the homotopy link as the the Hurewicz fibration associated to the induced map $X \to Y$.

We begin with the definition of the lifting property which characterizes homotopically stratified pairs. Let (X, Y) be a pair such that Y is a neighborhood retract of X, and fix an open neighborhood and a retraction $r : U \to Y$. Consider the following spaces:

$$\begin{aligned}
W_1(r) &= \{(x, \omega) \in Y \times \mathrm{Map}(I, Y) \mid x = \omega(1)\}, \\
W_2(r) &= \{(x, \omega) \in (U \setminus Y) \times \mathrm{Map}(I, Y) \mid r(x) = \omega(1)\}, \quad \text{and} \\
W(r) &= W_1(r) \cup W_2(r) = \{(x, \omega) \in U \times \mathrm{Map}(I, Y) \mid r(x) = \omega(1)\}.
\end{aligned}$$

Mapping spaces are always given the compact-open topology. Note that the map $w(r) : W(r) \to Y$ defined by $w(r)(x, \omega) = \omega(0)$ is the associated Hurewicz fibration of r, and $w(r)| : W_2(r) \to Y$ is the associated Hurewicz fibration of $r| : U \setminus Y \to Y$.

Definition 4.1. The pair (X, Y) has the $W(r)$-*lifting property* if there exists a map

$$\alpha : W(r) \to \mathrm{Map}(I, X)$$

such that

1. $\alpha(x, \omega)(0) = \omega(0)$ for all $(x, \omega) \in W(r)$,
2. $\alpha(x, \omega)(1) = x$ for all $(x, \omega) \in W(r)$,
3. if $(x, \omega) \in W_1(r)$, then $\alpha(x, \omega) = \omega$, and
4. if $(x, \omega) \in W_2(r)$, then $\alpha(x, \omega) \in \mathrm{Map}_s((I, 0), (X, Y)) = \mathrm{holink}(X, Y)$.

Theorem 4.2. *If X is a metric space and $Y \subseteq X$, then the following are equivalent*:

(i) *(X, Y) is homotopically stratified,*
(ii) *Y is a neighborhood retract of X and for every sufficiently small neighborhood U of Y and retraction $r : U \to Y$, (X, Y) has the $W(r)$-lifting property,*
(iii) *there exist a neighborhood U of Y and a retraction $r : U \to Y$ such that (X, Y) has the $W(r)$-lifting property.*

Proof. (i) *implies* (ii): Since (X, Y) is homotopically stratified, hence forward tame, there exists a neighborhood N of Y and a nearly strict deformation

$$H : (N \times I, Y \times I \cup N \times \{0\}) \to (X, Y).$$

In particular, Y is a neighborhood retract of X. Let U be any neighborhood of Y such that $U \subseteq N$ and let $r : U \to Y$ be any retraction. We will show that (X, Y) has the $W(r)$-lifting property. Define a map $\beta : W(r) \to \mathrm{Map}(I, Y)$ by the formula

$$\beta(x, \omega)(t) = \begin{cases} rH(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \omega(2 - 2t), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define $f : W_2(r) \to \mathrm{holink}(X, Y)$ by $f(x, \omega)(t) = H(x, t)$ for $t \in I$, and define

$$F : W_2(r) \times I \to Y$$

by $F(x, \omega, t) = \beta(x, \omega)(t)$ for $t \in I$. Note that we have a lifting problem:

$$\begin{array}{ccc} W_2(r) & \xrightarrow{f} & \mathrm{holink}(X, Y) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle q} \\ W_2(r) \times I & \xrightarrow{F} & Y. \end{array}$$

(Recall that q is evaluation at 0). Since part of our hypothesis is that q is a fibration, we have a solution $\tilde{F}$. We will use $\tilde{F}$ to define α, but to make sure that a certain extension to $W(r)$ is continuous on $W_1(r)$, we first need a lemma whose proof is postponed until later in this section.

Lemma 4.3. *There exists a map* $\gamma : W_2(r) \times I \to [0, 1]$ *such that*

(1) $\gamma(x, \omega, 0) = 1$ *for all* $(x, \omega) \in W_2(r)$,
(2) $\mathrm{diam}\{\tilde{F}(x, \omega, t)(s) \mid 0 \leq s \leq \gamma(x, \omega, t)\} \leq 2\,\mathrm{diam}\{\tilde{F}(x, \omega, 0)(s) \mid s \in I\}$ *for all* $(x, \omega, t) \in W_2(r) \times I$,
(3) $\gamma(x, \omega, t) = 0$ *if and only if* $t = 1$, *for all* $(x, \omega) \in W_2(r)$.

Assuming the lemma we complete the proof that (i) *implies* (ii) in Theorem 4.2. Define

$$\alpha : W_2(r) \to \mathrm{holink}(X, Y) \text{ by } \alpha(x, \omega)(t) = \tilde{F}(x, \omega, 1 - t)(\gamma(x, \omega, 1 - t)).$$

Then α extends to a map $\alpha : W(r) \to \mathrm{Map}(I, X)$ by setting $\alpha(x, \omega) = \omega$ for $(x, \omega) \in W_1(r)$. It is straightforward to verify that α is continuous and satisfies the condition of the $W(r)$-lifting property.
(ii) *implies* (iii) is obvious.
(iii) *implies* (i): Let $\alpha : W(r) \to \mathrm{Map}(I, X)$ satisfy the definition of the $W(r)$-lifting property where $r : U \to Y$ is some retraction of a neighborhood of Y. For each $x \in U$ let ω_x denote the constant path at $r(x)$. Define $H : U \times I \to X$ by

$$H(x, t) = \alpha(x, \omega_x)(t).$$

Then H is a nearly strict deformation of U into Y, so Y is forward tame in X. To see that $q : \mathrm{holink}(X, Y) \to Y$ is a fibration, consider a lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{f} & \mathrm{holink}(X, Y) \\ {\scriptstyle \times 0}\downarrow & & {\scriptstyle q}\downarrow \\ Z \times I & \xrightarrow{F} & Y. \end{array}$$

We may assume that Z is metric. Using a partition of unity one can construct a map $\epsilon : Z \to (0,1]$ such that for every $z \in Z$ and $0 \leq t \leq \epsilon(z)$, we have $f(z)(t) \in U$. Define a map $\omega : Z \times I \to \mathrm{Map}(I, Y)$ by

$$\omega(z,t)(s) = \begin{cases} F(z, t-2ts), & \text{if } 0 \leq s \leq 1/2 \\ r(f(z)(\epsilon(z)(2ts-t))), & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Note that $\omega(z,0)(s) = F(z,0) = f(z)(0)$ for all $z \in Z$ and $s \in I$. Now define

$$\delta : Z \times I \to \mathrm{Map}(I, X) \text{ by } \delta(z,t) = \alpha(f(z)(\epsilon(z)t), \omega(z,t))$$

and note that

(1) $\delta(z,0)(s) = F(z,0)$,
(2) $\delta(z,t)(1) = f(z)(\epsilon(z)t)$,
(3) $\delta(z,t)(0) = F(z,t)$.

Finally, define a solution $\tilde{F} : Z \times I \to \mathrm{holink}(X,Y)$ of the lifting problem by

$$\tilde{F}(z,t)(s) = \begin{cases} \delta(z,t)(s/\epsilon(z)t), & \text{if } 0 \leq s < \epsilon(z)t \\ f(z)(s), & \text{if } \epsilon(z)t \leq s \leq 1. \end{cases} \quad \square$$

Proof of Lemma 4.3. First note that $\{\tilde{F}(x,\omega,0)(s) \mid s \in I\} = \{H(x,s) \mid s \in I\}$ for each $(x,\omega) \in W_2(r)$. Now for $x \in U \setminus Y$, let $c(x) = \mathrm{diam}\{H(x,s) \mid s \in I\}$. Note that $0 < c(x)$. For each $(x,\omega,t) \in W_2(r) \times I$, let

$$\delta(x,\omega,t) = \mathrm{lub}\{s \in I | \, \mathrm{diam}\{\tilde{F}(x,\omega,t)(s') | 0 \leq s' \leq s\} \leq c(x)\}.$$

Note that $0 < \delta(x,\omega,t) \leq 1$. For each $(x,\omega,t) \in W_2(r) \times I$, let $V(x,\omega,t)$ be a neighborhood of (x,ω,t) such that whenever $(x',\omega',t') \in V(x,\omega,t)$, then

(1) $\mathrm{diam}\{\tilde{F}(x',\omega',t')(s) \mid 0 \leq s \leq \delta(x,\omega,t)\} < 3c(x)/2$, and
(2) $c(x) \leq 4c(x')/3$.

Let $\{V_\alpha\}$ be a locally finite refinement of $\{V(x,\omega,t)\}$ and let $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. For each α choose (x,ω,t) such that $V_\alpha \subseteq V(x,\omega,t)$ and set $\delta_\alpha = \delta(x,\omega,t)$. Define $\hat{\gamma} : W_2(r) \times I \to I$ by $\hat{\gamma} = \sum \delta_\alpha \phi_\alpha$. Clearly $\hat{\gamma}$ satisfies item (2) of the lemma, but we need to modify $\hat{\gamma}$ to achieve the other conditions. Using the paracompactness of $W_2(r)$, choose a neighborhood V of $W_2(r) \times \{0\}$ in $W_2(r) \times I$ such that if $(x,\omega,t) \in V$, then

$$\mathrm{diam}\{\tilde{F}(x,\omega,t)(s) \mid s \in I\} \leq 2c(x).$$

Let $\psi : W_2(r) \times I \to I$ be a map such that $\psi = 1$ on $W_2(r) \times \{0\}$, $\psi = 0$ off of V, and $\psi > 0$ on V. Finally set

$$\gamma(x,\omega,t) = (1-t)[(1-\psi(x,\omega,t))\hat{\gamma}(x,\omega,t) + \psi(x,\omega,t)]. \quad \square$$

Proposition 4.4. *If X is a metric space and $Y \subseteq X$, then the following are equivalent*:

(i) (X,Y) *is a homotopically stratified pair,*
(ii) *for every neighborhood U of Y in X, (U,Y) is a homotopically stratified pair,*
(iii) *there exists a neighborhood U of Y in X such that (U,Y) is a homotopically stratified pair.*

Proof. (i) *implies* (ii): Let U be a neighborhood of Y in X. Forward tameness implies there exist a neighborhood N of Y in X such that $N \subseteq U$ and a nearly strict deformation of N to Y in U which gives a retraction $r : N \to Y$. The proof of Theorem 4.2 (i) implies (ii) shows that if N is a sufficiently small neighborhood of Y in U, then (U,Y) has the $W(r)$–lifting property so that Theorem 4.2 may be invoked.
(ii) *implies* (iii) is obvious.
(iii) *implies* (i): By Theorem 4.2 we know that (U,Y) has the $W(r)$–lifting property for some r. It follows that (X,Y) has the $W(r)$–lifting property and Theorem 4.2 may be invoked once again. □

We now recall the definition of approximate fibrations as given in [29]. See [29, §12] for an explanation of how this definition relates to others in the literature.

Definition 4.5. A map $p : E \to B$ is an *approximate fibration* if for every commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times [0,1] & \xrightarrow{F} & B \end{array}$$

there is a controlled map $\tilde{F} : Z \times [0,1] \times [0,1) \to E$ from F to p such that $\tilde{F}(x,0,u) = f(x)$ for all $(x,u) \in Z \times [0,1)$. To say $\tilde{F}$ is a *controlled map* from F to p means the function $G : Z \times [0,1] \times [0,1] \to B$ defined by

$$G(z,t,u) = \begin{cases} p\tilde{F}(z,t,u), & \text{if } u < 1 \\ F(z,t), & \text{if } u = 1 \end{cases}$$

is continuous.

Lemma 4.6 (Open ended homotopies). *Suppose that $p : E \to B$ is an approximate fibration and that the following lifting problem is given*:

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times [0,1) & \xrightarrow{F} & B. \end{array}$$

Then there exists a controlled lift $\tilde{F}$, i.e., a map $\tilde{F} : Z \times [0,1) \times [0,1) \to E$ such that

(i) $\tilde{F}(z,0,u) = f(z)$ *for all $u \in [0,1)$, and*

(ii) *the function* $G : Z \times [0,1) \times [0,1] \to B$ *defined by*

$$G(z,t,u) = \begin{cases} p\tilde{F}(z,t,u), & \text{if } u < 1 \\ F(z,t), & \text{if } u = 1 \end{cases}$$

is continuous.

Proof. Let $\pi : \mathcal{E} \to B$ be the Hurewicz fibration associated to $p : E \to B$ and let $i : E \to \mathcal{E}$ be the inclusion. According to [29, 12.5] there is a controlled map $R : \mathcal{E} \times [0,1) \to E$ from π to p and a controlled homotopy $H : E \times [0,1] \times [0,1) \to E$ from id_e to Ri. This means that the function $\overline{R} : \mathcal{E} \times [0,1] \to B$ defined by

$$\overline{R}(x,t) = \begin{cases} pR(x,t), & \text{if } t < 1 \\ \pi(x), & \text{if } t = 1 \end{cases}$$

is continuous, that H satisfies $H(x,0,t) = x$ and $H(x,1,t) = R(i(x),t)$ for all $(X,t) \in E \times [0,1)$, and that the function $\overline{H} : E \times [0,1] \times [0,1] \to B$ defined by

$$\overline{H}(x,s,t) = \begin{cases} pH(x,s,t), & \text{if } t < 1 \\ p(x), & \text{if } t = 1 \end{cases}$$

is continuous. Given a lifting problem of the form

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times [0,1) & \xrightarrow{F} & B. \end{array}$$

there is an induced problem

$$\begin{array}{ccc} Z & \xrightarrow{if} & \mathcal{E} \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Z \times [0,1) & \xrightarrow{F} & B. \end{array}$$

Since π is a fibration, this second problem has an exact solution $\hat{F} : Z \times [0,1) \to \mathcal{E}$. Define $F' : Z \times [0,1) \times [0,1) \to E$ by $F'(z,s,t) = R(\hat{F}(z,s),t)$. Then a controlled solution $\widetilde{F} : Z \times [0,1) \times [0,1) \to E$ to the first problem can be defined by

$$\widetilde{F}(z,s,t) = \begin{cases} H(f(z), \frac{s}{1-t}, t), & \text{if } 0 \le s \le 1-t \\ F'(z, \frac{s-1-t}{t}, t), & \text{if } 1-t \le s \le 1. \end{cases}$$

One checks that the function G defined in the statement is continuous. □

Theorem 4.7. *If* X *and* Y *are metric spaces and* $p : X \to Y \times \mathbb{R}$ *is an approximate fibration, then the teardrop* $(X \cup_p Y, Y)$ *is a homotopically stratified pair.*

Proof. There exists a retraction $r : X \cup_p Y \to Y$ given by the composition

$$X \cup_p Y \xrightarrow{c} Y \times (-\infty, +\infty] \xrightarrow{\text{proj}} Y.$$

Since $X \cup_p Y$ is metric by Lemma 3.15, it suffices by Theorem 4.2 to show that $(X \cup_p Y, Y)$ has the $W(r)$–lifting property. We will first define α on $W_2(r)$ and then extend it to all of $W(r)$. To this end define

$$F : W_2(r) \times [0,1) \to Y \times \mathbb{R} \qquad \text{by} \qquad F(x,\omega,t) = (\omega(1-t), \frac{s}{1-t})$$

where s is defined by $p(x) = (r(x), s) \in Y \times \mathbb{R}$. Define $f : W_2(r) \to X$ by $f(x,\omega) = x$. Then we have a lifting problem

$$\begin{array}{ccc} W_2(r) & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ W_2(r) \times [0,1) & \xrightarrow{F} & Y \times \mathbb{R} \end{array}$$

to which we can apply Lemma 4.6 and get a controlled lift

$$\tilde{F} : W_2(r) \times [0,1) \times [0,1) \to X.$$

Let $G : W_2(r) \times [0,1) \times [0,1] \to Y \times \mathbb{R}$ be the map defined in Lemma 4.6. Using the paracompactness of $W_2(r) \times [0,1)$, there exists a map $\gamma : W_2(r) \times [0,1) \to [0,1)$ such that if $(x,\omega) \in W_2(r)$ and $1 - 1/i \le t \le 1 - 1/(i+1)$, then

$$\operatorname{diam} G(\{x,\omega,t\} \times [\gamma(x,\omega,t), 1]) < 1/i.$$

Then define $\hat{F} : W_2(r) \times [0,1] \to X \cup_p Y$ by

$$\hat{F}(x,\omega,t) = \begin{cases} \tilde{F}(x,\omega,\gamma(x,\omega,t)), & \text{if } 0 \le t < 1 \\ \omega(0), & \text{if } t = 1. \end{cases}$$

And define $\alpha : W_2(r) \to \operatorname{holink}(X \cup_p Y, Y)$ by

$$\alpha(x,\omega)(t) = \hat{F}(x,\omega,1-t).$$

Then α extends continuously to $\alpha : W(r) \to \operatorname{Map}(I, X \cup_p Y)$ by setting $\alpha(x,\omega) = \omega$ for $(x,\omega) \in W_1(r)$. $\square$

Proposition 4.8. *If X and Y are metric spaces and $p : X \to Y \times \mathbb{R}$ is an approximate fibration, then $q : \operatorname{holink}(X \cup_p Y, Y) \to Y$ is fibre homotopy equivalent to the Hurewicz fibration associated to the composition*

$$X \xrightarrow{p} Y \times \mathbb{R} \xrightarrow{\text{proj}} Y.$$

Proof. Let $r : X \cup_p Y \to Y$ be the retraction $X \cup_p Y \xrightarrow{c} Y \times (-\infty, +\infty] \xrightarrow{\text{proj}} Y$. Let $\pi = w(r)| : W_2(r) \to Y$ which is the Hurewicz fibration associated to $r| : X \to Y$. We must show that π is fibre homotopy equivalent to $q : \operatorname{holink}(X \cup_p Y, Y) \to Y$. It follows from the proof of Theorem 4.7 that $(X \cup_p Y, Y)$ has the $W(r)$–lifting property. Let $\alpha : W(r) \to \operatorname{Map}(I, X \cup_p Y)$ be a map as in Definition 4.1. Define $f : W_2(r) \to \operatorname{holink}(X \cup_p Y, Y)$ to be the restriction of α so that $f(x,\omega) = \alpha(x,\omega)$. We will show that f is a fibre homotopy equivalence with fibre homotopy inverse

$g : \mathrm{holink}(X \cup_p Y, Y) \to W_2(r)$ defined by $g(\omega) = (\omega(1), r\omega)$. We will define a fibre homotopy $G : gf \simeq \mathrm{id}_{W_2(r)}$ as follows. If $\omega \in \mathrm{Map}(I, Y)$ and $s \in I$, define $\omega_s^+ : I \to Y$ by $\omega_s^+(t) = \omega((1-s)t + s)$. Define a homotopy $E : W_2(r) \times I \to \mathrm{Map}(I, Y)$ by

$$E(x, \omega, s)(t) = \begin{cases} \omega(t), & \text{if } 0 \le t \le s \\ r\alpha(x, \omega_s^+)(\frac{t-s}{1-s}), & \text{if } s \le t < 1 \\ r(x), & \text{if } t = 1. \end{cases}$$

Then let $G((x, \omega), s) = (x, E(x, \omega, s))$. We will now define a fibre homotopy $F : \mathrm{id}_{\mathrm{holink}(X \cup_p Y, Y)} \simeq fg$ as follows. If $\omega \in \mathrm{holink}(X \cup_p Y, Y)$ and $s \in I$, define $\omega_s : I \to X \cup_p Y$ by $\omega_s(t) = \omega(ts)$. Then define F by

$$F(\omega, s)(t) = \begin{cases} \omega(0), & \text{if } t = 0 \\ \alpha(\omega(s), r\omega_s)(\frac{t}{s}), & \text{if } 0 < t \le s \\ \omega(t), & \text{if } s \le t \le 1. \end{cases} \quad \square$$

Lemma 4.9 (Folklore). *If $p : X \to Y$ is a proper approximate fibration between ANRs (locally compact, separable metric), then the homotopy fibre of p is finitely dominated.*

Proof. Fix a basepoint $y_0 \in Y$. The homotopy fibre of p is

$$W = \{(x, \omega) \in X \times Y^I \mid \omega(0) = p(x), \omega(1) = y_0\}.$$

Let U be an open neighborhood of y_0 which contracts to y_0 in Y; that is, there exists a homotopy $H : U \times I \to Y$ such that $H_0 = \text{inclusion} : U \to Y$, $H_1(U) = \{y_0\}$ and $H_t(y_0) = y_0$ for all $t \in I$. Let V be a compact neighborhood of y_0 such that $H(V \times I) \subseteq U$. It is well-known that for every open cover $\mathcal{U}$ of X there is a locally finite simplicial complex which $\mathcal{U}$-dominates X (see e.g. [42]). This fact together with the compactness of $p^{-1}(V)$ implies that there exist a locally finite simplicial complex L, maps $f : L \to X$, $g : X \to L$, and a homotopy $J : \mathrm{id}_X \simeq fg$ such that $J(p^{-1}(V) \times I) \subseteq p^{-1}(U)$. Note that $g(p^{-1}(V)) \subseteq f^{-1}(p^{-1}(U))$ and use the compactness of $p^{-1}(V)$ again to find a finite subcomplex K of L (in some fine triangulation) such that $g(p^{-1}(V)) \subseteq K$ and $f(K) \subseteq p^{-1}(U)$. We will show that K dominates W. Consider the lifting problem

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ W \times I & \xrightarrow{G} & Y \end{array}$$

where $G((x, \omega), t) = \omega(t)$ and $g(x, \omega) = x$. Since p is an approximate fibration there is an approximate solution $\widetilde{G} : W \times I \to X$. Assume that $p\widetilde{G}$ is so close to G that the image of $\widetilde{G}_1$ is in $p^{-1}(V)$ and that there is a homotopy $F : p\widetilde{G} \simeq G$ rel $W \times \{0\}$. Using the homotopy extension theorem we can insist that $F|W \times \{1\} \times I$ is given by $F((x, \omega), 1, s) = H(p\widetilde{G}_1(x, \omega), s)$. It follows that there is a homotopy $A : W \times I \times I \to Y$ such that

(1) $A((x, \omega), 0, s) = \omega(0)$,

(2) $A((x,\omega)1,s) = H(pfg\widetilde{G}_1(x,\omega),s)$,
(3) $A((x,\omega),t,1) = \omega(t)$,
(4) $A((x,\omega)t,0) = \begin{cases} p\widetilde{G}((x,\omega),2t), & 0 \leq t \leq \frac{1}{2} \\ pJ(\widetilde{G}_1(x,\omega),2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$

Define $d : K \to W$ and $u : W \to K$ by $d(x) = (f(x), H(pf(x),\cdot))$ and $u(x,\omega) = g(\widetilde{G}_1(x,\omega))$. The homotopy A can be used to construct a homotopy $du \simeq \mathrm{id}_W$. □

Corollary 4.10. *If X and Y are ANRs (locally compact, separable metric) and $p : X \to Y \times \mathbb{R}$ is a proper approximate fibration, then the (homotopy) fibre of $q : \mathrm{holink}(X \cup_p Y, Y) \to Y$ is finitely dominated. Moreover, $(X \cup_p Y, Y)$ is a homotopically stratified locally compact, separable metric pair with finitely dominated local holinks.*

Proof. It follows from Lemma 3.15 that $X \cup_p Y$ is metrizable. Since X and Y are separable, so is $X \cup_p Y$. Since p is proper, it follows easily that the teardrop collapse $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ is also proper (cf. the proof of Proposition 3.16). In particular, $X \cup_p Y$ is locally compact. By Theorem 4.7, $(X \cup_p Y, Y)$ is homotopically stratified. It follows from Proposition 4.8 that the homotopy fibre of $\mathrm{holink}(X \cup_p Y, Y) \to Y$ is homotopy equivalent to the homotopy fibre of p which is finitely dominated by Lemma 4.9. Thus, $(X \cup_p Y, Y)$ has finitely dominated local holinks. □

Corollary 4.11. *If B is a closed manifold and $p : M \to B \times \mathbb{R}$ is a manifold approximate fibration, then the teardrop $(M \cup_p B, B)$ is a manifold stratified pair.*

Proof. This follows immediately from Corollary 4.10. □

5. Spaces of stratified neighborhoods and manifold approximate fibrations

This section contains the details of the definitions of the simplicial set $\mathrm{MAF}^n(B \times \mathbb{R})$ of manifold approximate fibrations and the simplicial set $\mathrm{SN}^n(B)$ of stratified neighborhoods. Facts are established which are needed to define the simplicial map $\Psi : \mathrm{MAF}^n(B \times \mathbb{R}) \to \mathrm{SN}^n(B)$.

Definition 5.1. Suppose $A \times K$ is a closed subset of X and $\pi : X \to K$ is a map such that $\pi| : A \times K \to K$ is the projection.
(1) The pair $(X, A \times K)$ is a *sliced homotopically stratified pair* (*with respect to* π) if

- (i) $A \times K$ is sliced forward tame in X with respect to π,
- (ii) the evaluation $q : \mathrm{holink}_\pi(X, A \times K) \to A \times K$ is a fibration,
- (iii) (Local triviality near $A \times K$) there exist an open neighborhood W of $A \times K$ in X and a space U containing A such that for each $t \in K$ there exist an open neighborhood V of t in K and a f.p. open embedding $h : U \times V \to X$ such that $h| : A \times V \to X$ is the inclusion and $h(U \times V) = W \cap \pi^{-1}(V)$. That is, $\pi| : W \to K$ is a fibre bundle projection containing $A \times K \to K$ as a subbundle. In this case W is said to be a *locally trivial neighborhood of $A \times K$ in X*. If $V = K$, then W is said to be a *trivial neighborhood of $A \times K$ in X*.

(2) The pair $(X, A \times K)$ has *finitely dominated local holinks* (*with respect to* π) if the fibre of $q : \mathrm{holink}_\pi(X, A \times K) \to A \times K$ is finitely dominated.

(3) The pair $(X, A \times K)$ is a *sliced manifold stratified pair* (*with respect to* π) if it is a sliced homotopically stratified pair with finitely dominated local holinks, X is a locally compact separable metric space, A is a manifold, and for each $t \in K$ $\pi^{-1}(t) \setminus A \times \{t\}$ is a manifold.

Note that if K is contractible, then the local triviality condition near $A \times K$ implies that $A \times K$ has a trivial neighborhood in X.

Proposition 5.2. *Suppose $A \times K$ is a closed subset of a metric space X and $\pi : X \to K$ is a map such that $\pi| : A \times K \to K$ is the projection.*
(i) *If N is a neighborhood of $A \times K$ in X, then the inclusion* $\mathrm{holink}_\pi(N, A \times K) \to \mathrm{holink}_\pi(X, A \times K)$ *is a fibre homotopy equivalence from* $q : \mathrm{holink}_\pi(N, A \times K) \to A \times K$ *to* $q : \mathrm{holink}_\pi(X, A \times K) \to A \times K$.
(ii) *If N is a neighborhood of $A \times K$ in X, then* $q : \mathrm{holink}_\pi(X, A \times K) \to A \times K$ *is a fibration if and only if* $q : \mathrm{holink}_\pi(N, A \times K) \to A \times K$ *is.*
(iii) *If K is compact, the following are equivalent*:
  (a) *$(X, A \times K)$ is a sliced homotopically stratified pair,*
  (b) *for every neighborhood N of $A \times K$ in X, $(N, A \times K)$ is a homotopically stratified pair,*
  (c) *there exists a neighborhood N of $A \times K$ in X such that $(N, A \times K)$ is a homotopically stratified pair.*

(iv) *If N is a neighborhood of $A \times K$ in X, then $(X, A \times K)$ has finitely dominated local holinks if and only if $(N, A \times K)$ does.*
(v) *If K is compact and N is open an open neighborhood of $A \times K$ in X and $(X, A \times K)$ is a sliced manifold stratified pair, then so is $(N, A \times K)$.*

Proof. (i) (cf. [28, 1.12]) For each $\omega \in \mathrm{holink}_\pi(X, A \times K)$ choose a number $t_\omega \in (0, 1]$ such that $\omega([0, t_\omega]) \subseteq \mathrm{int}(N)$. Let $U(\omega)$ be an open neighborhood of ω in $\mathrm{holink}_\pi(X, A \times K)$ such that $\alpha([0, t_\omega]) \subseteq \mathrm{int}(N)$ for all $\alpha \in U(\omega)$. Since $\mathrm{holink}_\pi(X, A \times K)$ is a metric space, there is a locally finite refinement $\{U_i\}$ for the cover $\{U(\omega) \mid \omega \in \mathrm{holink}_\pi(X, A \times K)\}$ of $\mathrm{holink}_\pi(X, A \times K)$ and a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. For each i choose $\omega_i \in \mathrm{holink}_\pi(X, A \times K)$ such that $U_i \subseteq U(\omega_i)$ and let $t_i = t_{\omega_i}$. For each $\omega \in \mathrm{holink}_\pi(X, A \times K)$ let $m_\omega = \max\{t_i \mid \phi_i(\omega) \neq 0\}$. Note that $\omega([0, m_\omega]) \subseteq \mathrm{int}(N)$ and $\sum_i \phi_i(\omega) t_i \leq m_\omega$ for all ω. Define a homotopy $R : \mathrm{holink}_\pi(X, A \times K) \times I \to \mathrm{holink}_\pi(X, A \times K)$ by

$$R(\omega, t)(s) = \begin{cases} \omega(s), & \text{if } 0 \leq s \leq \sum_i \phi_i(\omega) t_i \\ \omega((1-t)s + t \sum_i \phi_i(\omega) t_i), & \text{if } \sum_i(\omega) t_i \leq s \leq 1. \end{cases}$$

Then R is a fibre deformation with $R_0 = \mathrm{id}$,

$$R_1(\mathrm{holink}_\pi(X, A \times K)) \subseteq \mathrm{holink}_\pi(N, A \times K)$$

and $R_t(\mathrm{holink}_\pi(N, A \times K)) \subseteq \mathrm{holink}_\pi(N, A \times K)$ for each t. The result follows immediately. Note also that if $\rho : \mathrm{holink}_\pi(X, A \times K) \to (0, 1]$ is defined by $\rho(\omega) = \sum_i \phi_i(\omega) t_i$, then ρ is continuous and $R_t(\omega)(s) = \omega(s)$ for all $0 \leq t \leq 1$ and $0 \leq s \leq \rho(\omega)$.
(ii) Let R and ρ be given as in the proof of (i). Suppose first that $q : \mathrm{holink}(N, A \times K) \to A \times K$ is a fibration. Then a homotopy lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{f} & \mathrm{holink}_\pi(X, A \times K) \\ \downarrow & & \downarrow q \\ Z \times I & \xrightarrow{F} & A \times K \end{array}$$

for $\text{holink}_\pi(X, A \times K) \to A \times K$ induces a problem

$$\begin{array}{ccc} Z & \xrightarrow{R_1 f} & \text{holink}_\pi(N, A \times K) \\ \Big\downarrow & & \Big\downarrow q \\ Z \times I & \xrightarrow{F} & A \times K \end{array}$$

for $\text{holink}_\pi(N, A\times K) \to A\times K$ which has a solution $G : Z\times I \to \text{holink}_\pi(N, A\times K)$. For each $\omega \in \text{holink}_\pi(X, A \times K)$ define

$$\tau_\omega : [0, \rho(\omega)] \times [0, 1] \to [0, 1] \times [0, \rho(\omega)] \ \text{ by } \ \tau_\omega(s, t) = (t - \frac{ts}{\rho(\omega)}, s).$$

Then a solution $\widetilde{F} : Z \times I \to \text{holink}_\pi(X, A \times K)$ of the original problem can be defined by

$$\widetilde{F}(z, t)(s) = \begin{cases} \widehat{G}(z, \tau_{f(z)}(s, t)), & \text{if } 0 \le s \le \rho(f(z)) \\ f(z)(s), & \text{if } \rho(f(z)) \le s \le 1 \end{cases}$$

where $\widehat{G}$ is the adjoint of G.
Conversely, suppose $q : \text{holink}(X, A \times K) \to A \times K$ is a fibration and N is a neighborhood of $A \times K$ in X. To show that $\text{holink}_\pi(N, A \times K) \to A \times K$ is a fibration, we may use the converse just proven to assume that N is open in X. Let

$$\begin{array}{ccc} Z & \xrightarrow{f} & \text{holink}_\pi(N, A \times K) \\ \Big\downarrow & & \Big\downarrow q \\ Z \times I & \xrightarrow{F} & A \times K \end{array}$$

be a homotopy lifting problem which by inclusion is also a problem for

$$\text{holink}_\pi(X, A \times K) \to A \times K.$$

Thus, there is a solution $G : Z\times I \to \text{holink}_\pi(X, A\times K)$ to this second problem. Let U be an open neighborhood of $Z\times\{0\}$ in $Z\times I$ such that $G(U) \subseteq \text{holink}_\pi(N, A\times K)$. Since it suffices to solve an universal problem, we may assume that Z is a metric space. Thus, there is a map $\sigma : Z \times I \to I$ such that $\sigma^{-1}(0) = Z \times \{0\}$ and $\sigma^{-1}(1) = (Z \times I) \setminus U$. Then $\widetilde{F} : Z \times I \to \text{holink}_\pi(N, A \times K)$ defined by $\widetilde{F}(z, t) = R(G(z, t), \sigma(z, t))$ is a solution of the original problem.
(iii) (a) *implies* (b): If N is a neighborhood of $A \times K$ in X, then $(N, A \times K)$ obviously satisfies the sliced forward tameness condition. From the fact that K is compact. it follows that $(N, A\times K)$ satisfies local triviality near $A\times K$. The holink fibration condition follows from (ii).
(b) *implies* (c) is obvious.
(c) *implies* (a): The sliced forward tameness and local triviality conditions obviously hold for $(X, A \times K)$ if they hold for $(N, A \times K)$. The holink fibration condition follows from (ii).
(iv) follows directly from (i).
(v) follows (iii) and (iv). $\square$

Lemma 5.3. *Suppose $A \times K$ is a closed subset of a space X and $\pi : X \to K$ is a map such that $\pi| : A \times K \to K$ is the projection. Let $f : K' \to K$ be a map and form the pull–back diagram*

$$\begin{array}{ccc} (X', A \times K') & \longrightarrow & (X, A \times K) \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ K' & \xrightarrow{f} & K. \end{array}$$

(i) *There is an induced pullback diagram*

$$\begin{array}{ccc} \operatorname{holink}_{\pi'}(X', A \times K') & \longrightarrow & \operatorname{holink}_{\pi}(X, A \times K) \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ A \times K' & \xrightarrow{\mathrm{id}_A \times f} & A \times K \end{array}$$

(ii) *If $(X, A \times K)$ is a sliced homotopically stratified pair, then so is $(X', A \times K')$.*
(iii) *If $(X, A \times K)$ has finitely dominated local holinks, then so does $(X', A \times K)$.*
(iv) *If $(X, A \times K)$ is a sliced manifold stratified pair, then so is $(X', A \times K')$.*

Proof. (i) and (ii) are elementary. The other parts follow immediately. $\square$

For the remainder of this section, B is an i-dimensional manifold without boundary together with a fixed embedding $B \subseteq \ell_2$ (of small capacity; e.g., we could take B to be inside of a finite dimensional subspace $\mathbb{R}^L$ of ℓ_2) and let $n \geq 5$ be a fixed integer.

Definition 5.4. The *space of stratified neighborhoods* of B is the simplicial set $\mathrm{SN}^n(B)$ whose k-simplices are subsets X of $\ell_2 \times \Delta^k$ of small capacity (see [29]) such that if $\pi : X \to \Delta^k$ is the restriction of the projection $\ell_2 \times \Delta^k \to \Delta^k$, then $(X, B \times \Delta^k)$ is a sliced manifold stratified pair with respect to π with $\dim(\pi^{-1}(t)) = n$ for each $t \in \Delta^k$.

We will denote a typical k–simplex of $\mathrm{SN}^n(B)$ by $\pi : (X, B \times \Delta^k) \to \Delta^k$ or, sometimes, just by $\pi : X \to \Delta^k$ and consider the embeddings $B \times \Delta^k \subseteq X$ and $X \subseteq \ell_2 \times \Delta^k$ understood. If $\pi : X \to \Delta^k$ is a k–simplex of $\mathrm{SN}^n(B)$, let $\partial X = \pi^{-1}(\partial \Delta^k)$ and let $\partial \pi = \pi| : \partial X \to \partial \Delta^k$, Thus $\partial \pi : \partial X \to \partial \Delta^k$ is a union of $k+1$ $(k-1)$–simplices of $\mathrm{SN}^n(B)$.

The following result characterizes the homotopy relation in $\mathrm{SN}^n(B)$. For notation, fix a base vertex of $\mathrm{SN}^n(B)$; that is, a manifold stratified pair (Y, B) with constant map $Y \to \Delta^0$. For each $k \geq 0$ the degenerate k–simplex on (Y, B) is the pair $(Y \times \Delta^k, B \times \Delta^k)$ with projection $Y \times \Delta^k \to \Delta^k$.

Proposition 5.5. *Let B be a closed manifold. Suppose $\pi : X \to \Delta^k$ and $\pi' : X' \to \Delta^k$ are two simplices of $\mathrm{SN}^n(B)$ such that $\partial \pi = \partial \pi' : \partial X = \partial X' = Y \times \partial \Delta^k \to \partial \Delta^k$ is the projection. The following are equivalent*:
(i) $\pi : X \to \Delta^k$ *and* $\pi' : X' \to \Delta^k$ *are homotopic* rel ∂,
(ii) *there exists a sliced manifold stratified pair $(W, B \times \Delta^k \times I)$ with map $\widetilde{\pi} : W \to \Delta^k \times I$ such that*

(1) $\widetilde{\pi}| = \pi : \widetilde{\pi}^{-1}(\Delta^k \times \{0\}) = X \to \Delta^k \times \{0\} = \Delta^k$,

(2) $\widetilde{\pi}| = \pi' : \widetilde{\pi}^{-1}(\Delta^k \times \{1\}) = X' \to \Delta^k \times \{1\} = \Delta^k$, *and*
(3) $\widetilde{\pi}| = \partial\pi \times \mathrm{id}_I = \partial\pi' \times \mathrm{id}_I = \ \mathrm{proj} : \partial X \times I = \partial X' \times I = Y \times \partial\Delta^k \times I \to \partial\Delta^k \times I$,

(iii) *there exist an open neighborhood U of $B \times \Delta^k$ in X and a f.p. open embedding $h : U \to X'$ such that $h| : (B \times \Delta^k) \cup (U \cap \partial X) \to (B \times \Delta^k) \cup (U \cap \partial X')$ is the identity.*

Proof. (i) *implies* (ii): Let $\widehat{\pi} : \widehat{W} \to \Delta^{k+1}$ be a homotopy rel ∂ from $\pi : X \to \Delta^k$ to $\pi : X' \to \Delta^k$ in $\mathrm{SN}^n(B)$. Thus, $\widehat{\pi} = \pi$ over $\partial_{k+1}\Delta^{k+1}$, $\widehat{\pi} = \pi'$ over $\partial_0\Delta^{k+1}$ and $\widehat{\pi}| = \ \mathrm{proj} : Y \times \partial_i\Delta^{k+1} \to \partial_i\Delta^{k+1}$ for $0 < i < k+1$. Consider the standard PL map $\rho : \Delta^k \times I \to \Delta^{k+1}$ such that $\rho^{-1}(\partial\Delta^{k+1}) = \partial(\Delta^k \times I)$ and ρ restricts to homeomorphisms $\Delta^k \times \{0\} \to \partial_{k+1}\Delta^{k+1}$ and $\Delta^k \times \{1\} \to \partial_0\Delta^{k+1}$. Form the pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{\widetilde{\pi}} & \Delta^k \times I \\ \big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \widehat{W} & \xrightarrow{\widehat{\pi}} & \Delta^{k+1} \end{array}$$

It follows from Lemma 5.3(iv) that $(W, B \times \Delta^k \times I)$ is a sliced manifold stratified pair with map $\widetilde{\pi}$.
(ii) *implies* (iii): Let V be an open neighborhood of $B \times \Delta^k \times I$ in W such that $\widetilde{\pi}| : V \to \Delta^k \times I$ is a (trivial) fibre bundle projection containing $B \times \Delta^k \times I \to \Delta^k \times I$ as a subbundle. Choose an open neighborhood U of $B \times \Delta^k \times \{0\}$ in $V \cap \widetilde{\pi}^{-1}(\Delta^k \times \{0\}) = X$ such that

$$[U \cap \widetilde{\pi}^{-1}(\partial\Delta^k \times \{0\})] \times I \subseteq V \cap \widetilde{\pi}^{-1}(\partial\Delta^k \times I) \subseteq \partial X \times I = \partial X' \times I.$$

Let $J = (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times I) \subseteq \Delta^k \times I$ and choose a homeomorphism $\alpha : J \times I \to \Delta^k \times I$ such that $\alpha| : J \times \{0\} \to J$ is the identity. Since $\widetilde{\pi}| : V \to \Delta^k \times I$ is trivial, there exists a homeomorphism $g : [\widetilde{\pi}^{-1}(J) \cap V] \times I \to V$ such that

$$\begin{array}{ccc} [\widetilde{\pi}^{-1}(J) \cap V] \times I & \xrightarrow{g} & V \\ {\scriptstyle \widetilde{\pi} \times \mathrm{id}_I}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi}|}\ V \\ J \times I & \xrightarrow{\alpha} & \Delta^k \times I \end{array}$$

commutes, $g|B \times J \times I$ equals $\mathrm{id}_B \times \alpha : B \times J \times I \to B \times \Delta^k \times I \subseteq V$, and $g| : [\widetilde{\pi}^{-1}(J) \cap V] \times \{0\} \to \widetilde{\pi}^{-1}(J) \cap V$ is the identity. Define $h : U \to \widetilde{\pi}^{-1}(\Delta^k \times \{1\}) = X'$ by setting $h(x) = g(x, 1)$ for all $x \in U$.
(iii) *implies* (i): Let N be a compact neighborhood of $B \times \Delta^k$ in X such that $N \subseteq U$. By the small capacity assumption, there exists a f.p. isotopy $H_t : \ell_2 \times \Delta^k \to \ell_2 \times \Delta^k$, $0 \le t \le 1$, such that $H_0 = \mathrm{id}_{\ell_2}$, $H_t|(B \times \Delta^k) \cup \ell_2 \times \partial\Delta^k$ is the identity for each $t \in I$, and $H_1|N = h|N : N \to X' \subseteq \ell_2 \times \Delta^k$. Let

$$W = (\partial X \times I) \cup (X \times \{0\}) \cup (X' \times \{1\}) \cup \{(H_t(x), t) \mid x \in \mathrm{int}(N), t \in I\}.$$

Proposition 5.2 implies that $(N \times I, B \times \Delta^k \times I)$ is a sliced homotopically stratified pair with finitely dominated local holinks, which in turn implies that $(W, B \times \Delta^k \times I)$ is a sliced manifold stratified pair. Now W induces a sliced manifold stratified pair $(\widehat{W}, B \times \Delta^{k+1})$ such that W is the pullback of $\widehat{W}$ along the map $\rho : \Delta^k \times I \to \Delta^{k+1}$ of (i), and $(\widehat{W}, B \times \Delta^{k+1})$ is the desired homotopy from X to X' rel ∂. $\square$

The next result follows from Proposition 5.5 by setting $k = 0$.

Corollary 5.6. *Let B be a closed manifold. Two vertices (X,B), (X',B) are in the same component of* $\mathrm{SN}^n(B)$ *if and only if they are germ equivalent*; *that is, there exist an open neighborhood U of B in X and an open embedding $h : U \to X'$ such that $h| : B \to X'$ is the inclusion.* $\square$

In order for homotopy theory to work well on the space of stratified neighborhoods, we need the following observation.

Proposition 5.7. $\mathrm{SN}^n(B)$ *satisfies the Kan condition.*

Proof. Suppose there is a collection of $k{+}1$ k–simplices $(X_j, B\times\partial_j\Delta^{k+1})$ of $\mathrm{SN}^n(B)$, $j=0,1,\dots,i-1,i+1,\dots,k+1$, which satisfy the compatibility condition (see [41, p. 2]). For $X=\cup X_j$ there is a natural map $X\to B\times w_i\Delta^{k+1}$ where $w_i\Delta^{k+1}$ is the union of all k–dimensional faces of Δ^{k+1} save $\partial_i\Delta^{k+1}$. It is elementary to verify that $(X, B\times w_i\Delta^{k+1})$ is a sliced manifold stratified pair. A possible exception is in the verification of the holink fibration condition, but that condition follows from [29, 16.2]. Pulling back along a retraction $\Delta^{k+1}\to w_i\Delta^{k+1}$ gives (by Lemma 5.3) a sliced manifold stratified pair $(\widetilde{X}, B\times\Delta^{k+1})$ which is the required $(k+1)$–simplex of $\mathrm{SN}^n(B)$. $\square$

Now recall the following definition from [29].

Definition 5.8. The *space of manifold approximate fibrations over $B\times\mathbb{R}$* is the simplicial set $\mathrm{MAF}^n(B\times\mathbb{R})$ whose k-simplices are subsets M of $\ell_2\times B\times\mathbb{R}\times\Delta^k$ of small capacity such that

(i) the restriction of projection $M\to\Delta^k$ is a fibre bundle projection with fibres n-dimensional manifolds without boundary. Let M_t denote the fibre over $t\in\Delta^k$.

(ii) the restriction of projection $p : M\to B\times\mathbb{R}\times\Delta^k$ has the property that $p_t = p| : M_t\to B\times\mathbb{R}\times\{t\}$ is a manifold approximate fibration for each $t\in\Delta^k$.

We will denote a typical k–simplex of $\mathrm{MAF}^n(B\times\mathbb{R})$ by $p : M\to B\times\mathbb{R}\times\Delta^k$ and consider the embeddings $B\times\Delta^k\subseteq X$ and $X\subseteq\ell_2\times\Delta^k$ understood.

Definition of $\Psi : \mathrm{MAF}^n(B{\times}\mathbb{R})\to\mathrm{SN}^n(B)$. It will be convenient to fix a teardrop of B in ℓ_2 which contains all the teardrops constructed form $\mathrm{MAF}^n(B\times\mathbb{R})$. To this end let

$$\mu : \ell_2\times B\times\mathbb{R}\to B\times\mathbb{R}$$

denote projection and let

$$T(B) = (\ell_2\times B\times\mathbb{R})\cup_\mu B$$

be the teardrop of μ. It follows from Lemma 4.3 that $T(B)$ is metrizable. Since B is separable, $T(B)$ is also separable. Hence, $T(B)$ embeds in ℓ_2 and we fix an embedding $T(B)\subseteq\ell_2$ of small capacity such that $B\subseteq T(B)\subset\ell_2$ is the original fixed embedding $B\subseteq\ell_2$.

We now define the simplicial map $\Psi : \mathrm{MAF}^n(B\times\mathbb{R})\to\mathrm{SN}^n(B)$. Given a k-simplex $M\subseteq\ell_2\times B\times\mathbb{R}\times\Delta^k$ of $\mathrm{MAF}^n(B\times\mathbb{R})$, we get a commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{\subseteq} & \ell_2\times B\times\mathbb{R}\times\Delta^k \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mu\times\mathrm{id}_{\Delta^k}} \\ B\times\mathbb{R}\times\Delta^k & \xrightarrow{=} & B\times\mathbb{R}\times\Delta^k \end{array}$$

Thus, $M \cup_p (B \times \Delta^k) \subseteq (\ell_2 \times B \times \mathbb{R} \times \Delta^k) \cup_{\mu \times \mathrm{id}_{\Delta^k}} (B \times \Delta^k) = T(B) \times \Delta^k \subseteq \ell_2 \times \Delta^k$. It will be shown below that $(M \cup_p (B \times \Delta^k), B \times \Delta^k)$ is a k-simplex of $\mathrm{SN}^n(B)$, and so we set $\Psi(M) = (M \cup_p (B \times \Delta^k), B \times \Delta^k)$.

Proof that $\Psi(M)$ is a k–simplex of $\mathrm{SN}^n(B)$. It is clear from the construction that $M \cup_p (B \times \Delta^k)$ is a subset of $\ell_2 \times \Delta^k$ of small capacity. Since each $p_t : M_t \to B \times \mathbb{R} \times \{t\}$ is a manifold approximate fibration, it follows from Corollary 4.11 that $(M_t \cup_{p_t} B \times \{t\}, B \times \{t\})$ is a manifold stratified pair for each $t \in \Delta^k$. Therefore, the sliced forward tameness, holink fibration and finitely dominated local holinks conditions follow from Claim 5.9 and Lemmas 5.10 and 5.11 below. To verify the local triviality condition let $\mathcal{U}$ be the open cover of $B \times \mathbb{R}$ consisting of all sets of the form

$$B(x, \frac{1}{|y| + |}) \times (y - \frac{1}{|y|+1}, y + \frac{1}{|y|+1})$$

where $(x, y) \in B \times \mathbb{R}$ and $B(x, r)$ denotes the ball about x in B of radius r. The point is that the diameters of members of $\mathcal{U}$ are small near $B \times \{+\infty\}$ and there is a maximum diameter. By [24] there is a homeomorphism $H : M \times \Delta^k \to M \times \Delta^k$ such that H is fibre preserving over Δ^k, $H_0 = id$, and pH is $\mathcal{U} \times \Delta^k$-close to $p_0 \times \mathrm{id}_{\Delta^k}$. The local triviality condition follows from the following claim and the fact that $(M \cup_{p_0} B) \times \Delta^k = (M \times \Delta^k) \cup_{p_0 \times \mathrm{id}_{\Delta^k}} (B \times \Delta^k)$.

Claim 5.9. *The map $h : (M \times \Delta^k) \cup_{p_0 \times \mathrm{id}_{\Delta^k}} (B \times \Delta^k) \to (M \times \Delta^k) \cup_p (B \times \Delta^k)$, defined by $h| : M \times \Delta^k \to M \times \Delta^k$ is H and $h| : B \times \Delta^k \to B \times \Delta^k$ is the identity, is a homeomorphism.*

Proof. We show that the map

$$g : (M \times \Delta^k) \cup_{p_0 \times \mathrm{id}_{\Delta^k}} (B \times \Delta^k) \xrightarrow{h} (M \times \Delta^k) \cup_p (B \times \Delta^k) \xrightarrow{c} B \times (-\infty, +\infty] \times \Delta^k$$

is continuous with c the teardrop collapse for p. For this it suffices to show that if $(x_n, t_n) \in M \times \Delta^k$, $(b, t) \in B \times \Delta^k$ and $(x_n, t_n) \to (b, t)$ in $(M \times \Delta^k) \cup_{p_0 \times \mathrm{id}} (B \times \Delta^k)$, then $g(x_n, t_n) \to (b, +\infty, t)$ in $B \times (-\infty, +\infty] \times \Delta^k$. Let $c' : (M \times \Delta^k) \cup_{p_0 \times \mathrm{id}} (B \times \Delta^k) \to B \times (-\infty, +\infty] \times \Delta^k$ be the collapse. Since c' is continuous, $c'(x_n, t_n) \to (b, +\infty, t)$ and so $(p_0(x_n), t_n) \to (b, +\infty, t)$. Given $\epsilon > 0$ there exists an integer K such that if $U \in \mathcal{U}$ meets $B \times [K, +\infty)$, then $\operatorname{diam} U < \epsilon$. There exists a positive integer M such that if $n \geq M$, then $p_0(x_n) \in B \times [K, +\infty)$ and $(p_0(x_n), t_n)$ is ϵ-close to $(b, +\infty, t)$. Now suppose $n \geq M$ and consider $g(x_n, t_n)$. Note that $g(x_n, t_n) = pH(x_n, t_n)$. There exists $U \in \mathcal{U}$ such that $pH(x_n, t_n)$ and $(p_0(x_n), t_n)$ are both in $U \times \Delta^k$; i.e., $p_{t_n} H_{t_n}(x_n)$, $p_0(x_n) \in \mathcal{U}$. Since $p_0(x_n) \in B \times [K, +\infty), \operatorname{diam} U < \epsilon$. Thus, $pH(x_n, t_n)$ and $(p_0(x_n), t_n)$ are ϵ-close measured in $B \times \mathbb{R} \times \Delta^k$. Since $(p_0(x_n), t_n)$ is ϵ-close to $(b, +\infty, t)$, we have shown that $g(x_n, t_n)$ is ϵ'-close to $(b, +\infty, t)$ where $\epsilon' > 0$ is small if ϵ is. Thus, g is continuous. This shows h is continuous by Lemma 3.4. Since p is also $\mathcal{U} \times \Delta^k$-close to $(p_0 \times \Delta^k)H^{-1}$, a similar proof shows that h^{-1} is continuous. □

We finish this section with the two lemmas mentioned above.

Lemma 5.10. *Suppose B is forward tame in X.*

(i) *If Y is any space, then $B \times Y$ is sliced forward tame in $X \times Y$ with respect to projection $X \times Y \to Y$.*

(ii) *If $\pi : E \to Y$ is a map of spaces and $h : X \times Y \to E$ is a homeomorphism such that πh is projection, then $h(B \times Y)$ is sliced forward tame in E with respect to π.*

Proof. (i) is obvious, and (ii) follows from (i) by using a sliced nearly strict deformation in $X \times Y$ conjugated with h. □

Lemma 5.11. *Suppose $B \subseteq X$ and* $\mathrm{holink}(X, B) \to B$ *is a fibration.*

(i) *If Y is any space, then* $\mathrm{holink}_{p_2}(X \times Y, B \times Y) \to B \times Y$ *is a fibration where p_2 is second coordinate projection.*

(ii) *If $\pi : E \to Y$ is a map of spaces and $h : X \times Y \to E$ is a homeomorphism such that πh is projection, then* $\mathrm{holink}_\pi(E, h(B \times Y)) \to h(B \times Y)$ *is a fibration.*

Proof. For (i) note that we have the following commuting diagram where $\nu(\omega) = (\omega', p_2\omega(0))$ and ω' is $[0, 1] \xrightarrow{\omega} X \times Y \xrightarrow{\mathrm{proj}} X$:

$$\begin{array}{ccc}
\mathrm{holink}_{p_2}(X \times Y, B \times Y) & \xrightarrow{\nu} & \mathrm{holink}(X, B) \times Y \\
\downarrow & & \downarrow \\
B \times Y & \xrightarrow{=} & B \times Y.
\end{array}$$

For (ii) note that we have the following commuting diagram where λ is the homeomorphism defined by $\lambda(\omega) = h \circ \omega$:

$$\begin{array}{ccc}
\mathrm{holink}_{p_2}(X \times Y, B \times Y) & \xrightarrow{\lambda} & \mathrm{holink}_\pi(E, h(B \times Y)) \\
\downarrow & & \downarrow \\
B \times Y & \xrightarrow{h} & h(B \times Y).
\end{array}$$ □

6. Homotopy near the lower stratum

The main theorems of this paper on Teardrop Neighborhood Existence (2.1) and Neighborhood Germ Classification (2.2, 2.3) have two aspects in their proofs: homotopy theoretic and manifold theoretic. This is already evident in §4 if one compares Theorem 4.7, which says that the teardrop of an approximate fibration is a homotopically stratified pair, with Corollary 4.11 which says that the teardrop of a manifold approximate fibration is a manifold stratified pair. This section contains the homotopy theoretic part of the remaining aspects of this paper's main existence and classification theorems. The main result here, Theorem 6.8, produces from a homotopically stratified pair (X, A) with finitely dominated local holinks, a $\mathcal{U}$–fibration over $A \times (0, +\infty)$ for arbitrarily small open covers $\mathcal{U}$ of $A \times (0, +\infty)$ (outside the setting of manifolds this is not quite the same notion as an approximate fibration). The proof involves showing that the mapping cylinder of the holink evaluation is a good homotopy model for a neighborhood germ of A in X. The idea of a good homotopy model is made precise with the notion of a 'strong $\mathcal{U}$–homotopy equivalence near A' in Definition 6.1.

There are three main steps to the proof of 6.8 corresponding to the three main hypotheses: holink evaluation is a fibration, forward tameness and finitely dominated

local holinks. The first step is Proposition 6.3 which shows how being modelled on the mapping cylinder of a fibration yields $\mathcal{U}$–fibrations (we apply this to the holink evaluation fibration). The second step, Proposition 6.5, shows that forward tameness is enough to get started in showing that the mapping cylinder of holink evaluation is a good model for a neighborhood of A in X. Finally, the third step, Proposition 6.7, adds the finitely dominated local holinks condition to produce the strong $\mathcal{U}$–homotopy equivalence near A. Of course, all of this must be done sliced (or fibre preserving) over Δ^k in order to obtain the Higher Classification Theorem 2.3.

We begin with the following definition of strong homotopy equivalences near A.

Definition 6.1. Suppose X_1 and X_2 are spaces containing $A \times \Delta^k$ with maps $\pi_i : X_i \to \Delta^k$ such that $\pi_i| : A \times \Delta^k \to \Delta^k$ is projection for $i = 1, 2$. Suppose $p : X_2 \to A \times (-\infty, +\infty] \times \Delta^k$ is a map which is fibre preserving over Δ^k and such that $p^{-1}(A \times \{+\infty\} \times \Delta^k) = A \times \Delta^k$ and $p| : A \times \Delta^k \to A \times \{+\infty\} \times \Delta^k$ is the identity. Suppose $\mathcal{U}$ is an open cover of $A \times \mathbb{R} \times \Delta^k$. A *strong f.p. $\mathcal{U}$–homotopy equivalence near $A \times \Delta^k$*

$$(f, g, X_1', X_2') : X_1 \to X_2$$

is defined by maps

$$f : X_1' \to X_2, \qquad g : X_2' \to X_1$$

such that

(i) X_1' a closed neighborhood of $A\times\Delta^k$ in X_1 and $X_2' = p^{-1}(A\times[t_2, +\infty]\times\Delta^k)$ for some $t_2 \in \mathbb{R}$,

(ii) the maps

$$f : (X_1', A \times \Delta^k) \to (X_2, A \times \Delta^k),$$
$$g : (X_2', A \times \Delta^k) \to (X_1, A \times \Delta^k)$$

are fibre preserving over Δ^k, strict and the identity on $A \times \Delta^k$,

together with homotopies

$$F : gf| \simeq \text{inclusion} : f^{-1}(X_2') \to X_1,$$
$$G : fg| \simeq \text{inclusion} : g^{-1}(X_1') \to X_2$$

such that

(iii) F, G are fibre preserving over Δ^k, rel $A \times \Delta^k$, and strict as homotopies between pairs $(f^{-1}(X_2'), A \times \Delta^k) \times I \to (X_1, A \times \Delta^k)$ and $(g^{-1}(X_1'), A \times \Delta^k) \times I \to (X_2, A \times \Delta^k)$,

(iv) for every $x \in f^{-1}(X_2') \setminus (A \times \Delta^k)$ with $\{x\} \times I \subseteq F^{-1}(X_1')$ there exists $U \in \mathcal{U}$ such that $pfF(\{x\} \times I) \subseteq U$,

(v) for every $x \in g^{-1}(X_1') \setminus (A \times \Delta^k)$ there exists $U \in \mathcal{U}$ such that $pG(\{x\} \times I) \subseteq U$.

Sliced homotopy lifting properties are just the parametric versions of ordinary lifting properties. These are used to define sliced $\mathcal{U}$–fibrations, sliced approximate fibrations and sliced manifold approximate fibrations (see [22]). We include the following definition for completeness.

Definition 6.2. Suppose $p : E \to A \times \Delta$ is a map (with Δ playing the role of the parameter space), $V \subseteq A \times \Delta$ and $\mathcal{U}$ is an open cover of $A \times \Delta$. Then P is a *sliced $\mathcal{U}$–fibration over V* if for every commuting diagram of maps which are f.p. over Δ

$$\begin{array}{ccc}
Z \times \Delta & \xrightarrow{f} & E \\
{\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\
Z \times \Delta \times I & \xrightarrow{F} & A \times \Delta
\end{array}$$

with $\mathrm{Im}(F) \subseteq V$, there exists an f.p. (over Δ) map $\tilde{F} : Z \times \Delta \times I \to E$ such that $\tilde{F}_0 = f$ and $p\tilde{F}$ is $\mathcal{U}$–close to F. If $V = A \times \Delta$, then p is a *sliced $\mathcal{U}$–fibration.* If p is a sliced $\mathcal{U}$–fibration for every open cover $\mathcal{U}$, then p is a *sliced approximate fibration.* If $E \to \Delta$ is a fibre bundle projection with manifold fibres (without boundary), A is a manifold (without boundary) and p is a proper sliced approximate fibration, then p is said to be a *sliced manifold approximate fibration.*

A map $p : E \to A$ is *proper over a subspace* $V \subseteq A$ if for every compact subspace $K \subseteq V$, $p^{-1}(K)$ is compact. We do not insist that proper maps be onto.

The following result shows that it is significant to be strongly f.p. $\mathcal{U}$–homotopy equivalent to the mapping cylinder of a fibration near the base of the mapping cylinder.

Proposition 6.3. *Suppose $q : E \to A \times \Delta^k$ is a fibration and $Q : \overset{\circ}{\mathrm{cyl}}(q) \to A \times (-\infty, +\infty] \times \Delta^k$ is the teardrop collapse. Suppose X is a locally compact separable metric space containing $A \times \Delta^k$ with a map $\pi : X \to \Delta^k$ such that $\pi| : A \times \Delta^k \to \Delta^k$ is projection and $\mathcal{U}$ is an open cover of $A \times \mathbb{R} \times \Delta^k$. Suppose $(f, g, X_1', X_2') : X \to \overset{\circ}{\mathrm{cyl}}(q)$ is a strong f.p. $\mathcal{U}$–homotopy equivalence near $A \times \Delta^k$ and $Qf : X_1' \to A \times (-\infty, +\infty] \times \Delta^k$ is proper. Then there exists an open neighborhood V of $A \times \{+\infty\} \times \Delta^k$ in $A \times (-\infty, +\infty] \times \Delta^k$ such that $Qf : X_1' \to A \times (-\infty, +\infty] \times \Delta^k$ is a sliced $\mathrm{st}^2(\mathcal{U})$–fibration over $(A \times \mathbb{R} \times \Delta^k) \cap V$.*

Proof. If $X_2' = Q^{-1}(A \times [t_2, +\infty] \times \Delta^k)$ choose an open neighborhood V of $A \times \{+\infty\} \times \Delta^k$ in $A \times (-\infty, +\infty] \times \Delta^k$ such that

(i) $V \subseteq A \times [t_2, +\infty] \times \Delta^k$,
(ii) $Q^{-1}(V) \subseteq g^{-1}(X_1')$ (this is possible since Q is a closed map over $A \times \Delta^k$),
(iii) $(Qf)^{-1}(V) \subseteq f^{-1}(X_2')$ (this is possible since Qf is proper, hence a closed map), and
(iv) $(Qf)^{-1}(V) \times I \subseteq F^{-1}(X_1')$ (this is possible since Qf is proper and F is the identity on $A \times \Delta^k \times I$).

A sliced homotopy lifting problem

$$\begin{array}{ccc}
Z \times \Delta^k & \xrightarrow{d} & X_1' \\
{\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle Qf} \\
Z \times \Delta^k \times I & \xrightarrow{D} & A \times (-\infty, +\infty] \times \Delta^k
\end{array}$$

with $\mathrm{Im}(D) \subseteq (A \times \mathbb{R} \times \Delta^k) \cap V$ yields another lifting problem

$$\begin{array}{ccc} Z \times \Delta^k & \xrightarrow{fd} & \overset{\circ}{\mathrm{cyl}}(q) \setminus (A \times \Delta^k) \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle Q|} \\ Z \times \Delta^k \times I & \xrightarrow{D} & A \times (-\infty, +\infty] \times \Delta^k. \end{array}$$

Since $\overset{\circ}{\mathrm{cyl}}(q) \setminus (A \times \Delta^k) = E \times \mathbb{R}$ and $Q| = q \times \mathrm{id}_{\mathbb{R}}$ is a fibration, this second problem has an exact solution $\widetilde{D}^1 : Z \times \Delta^k \times I \to E \times \mathbb{R}$ (so that $\widetilde{D}^1 | Z \times \Delta^k \times \{0\} = fd$ and $(q \times \mathrm{id}_{\mathbb{R}})\widetilde{D}^1 = D$). By choice of V, $\mathrm{Im}(\widetilde{D}^1) \subseteq X'_2$ and $\mathrm{Im}(g\widetilde{D}^1) \subseteq X'_1$. Define $\widetilde{D}^2 = g\widetilde{D}^1 : Z \times \Delta^k \times I \to X'_1$ and note that $Qf\widetilde{D}^2 = Qfg\widetilde{D}^1$ is $\mathcal{U}$–close to $Q\widetilde{D}^1 = D$. Except for the fact that $\widetilde{D}^2 | Z \times \Delta^k \times \{0\}$ need not equal d, $\widetilde{D}^2$ would be an approximate solution to the original problem. However, $\widetilde{D}^2 | Z \times \Delta^k \times \{0\} = g\widetilde{D}^1| = gfd$ and gfd is $(Qf)^{-1}(\mathcal{U})$–homotopic to d. Thus a standard argument using paracompactness allows a modification of $\widetilde{D}^2$ to get a $\mathrm{st}^2(\mathcal{U})$–solution $\widetilde{D} : Z \times \Delta^k \times I \to X'_1$ (see [28, Prop. 16.3]). $\square$

Notation 6.4. For the remainder of this section suppose $A \times \Delta^k \subseteq X$ and $\pi : X \to \Delta^k$ is a map such that $\pi| : A \times \Delta^k \to \Delta^k$ is the projection and $q : \mathrm{holink}_\pi(X, A \times \Delta^k) \to A \times \Delta^k$ is the evaluation. The open mapping cylinder of q is identified with the teardrop

$$\overset{\circ}{\mathrm{cyl}}(q) = (\mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R}) \cup_{q \times \mathrm{id}} (A \times \Delta^k)$$

where $q \times \mathrm{id} : \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \to A \times \mathbb{R} \times \Delta^k$. Let $Q : \overset{\circ}{\mathrm{cyl}}(q) \to A \times (-\infty, +\infty] \times \Delta^k$ be the teardrop collapse.

The genesis of the ideas in the next two results is in [24, 4.7] and [48, 2.4]. See especially [28, 9.13, 9.14].

Proposition 6.5. *Suppose X is a locally compact separable metric space, A is compact and $A \times \Delta^k$ is sliced forward tame in X with respect to π. Then there exist a compact neighborhood Y of $A \times \Delta^k$ in X and maps*

$$f : Y \to \overset{\circ}{\mathrm{cyl}}(q), \qquad g : \overset{\circ}{\mathrm{cyl}}(q) \to Y$$

together with homotopies

$$F : igf \simeq i : Y \to X, \qquad G : fg \simeq \mathrm{id} : \overset{\circ}{\mathrm{cyl}}(q) \to \overset{\circ}{\mathrm{cyl}}(q)$$

with $i : Y \to X$ the inclusion such that

(i) *f, g, F, G are* rel $A \times \Delta^k$*,*
(ii) *f, g, F, G are f.p. over Δ^k,*
(iii) *f, g, F, G are strict maps or homotopies between the pairs $(X, A \times \Delta^k)$ and $(\overset{\circ}{\mathrm{cyl}}(q), A \times \Delta^k)$,*
(iv) *for every $N \geq 0$ there exists $M \geq 0$ such that*

$$(Qfg)^{-1}(A \times (-\infty, N] \times \Delta^k) \subseteq Q^{-1}(A \times (-\infty, M] \times \Delta^k),$$

(v) *for every $N \geq 0$ there exists $M \geq 0$ such that*

$$G(Q^{-1}(A \times [M, +\infty] \times \Delta^k) \times I) \subseteq Q^{-1}(A \times [N, +\infty] \times \Delta^k).$$

Proof. (cf. [28,9.13].) Let d be a metric for X and let Y be a compact neighborhood of $A \times \Delta^k$ in X for which there exists a nearly strict deformation $H : (Y \times I, A \times \Delta^k \times I \cup Y \times \{0\}) \to (X, A \times \Delta^k)$ of Y into $A \times \Delta^k$ which is f.p. over Δ^k. It is easy to modify H so that it has the additional property that if $N = 1, 2, 3, \ldots$ and $x \in H(Y \times [0, 1/N])$, then $d(x, A) \leq 1/N$. Let $\hat{H} : Y \setminus (A \times \Delta^k) \to \mathrm{holink}_\pi(X, A \times \Delta^k)$ be the adjoint of H. Choose a compact neighborhood Y' of $A \times \Delta^k$ in X such that $Y' \subseteq Y$ and $\hat{H}(Y') \subseteq \mathrm{holink}_\pi(Y, A \times \Delta^k)$. Use i also to denote the inclusion $i : Y' \to X$. From Proposition 5.2(i), it induces a fibre homotopy equivalence $i_* : \mathrm{holink}_\pi(Y', A \times \Delta^k) \to \mathrm{holink}_\pi(X, A \times \Delta^k)$. Let $R : \mathrm{holink}_\pi(X, A \times \Delta^k) \times I \to \mathrm{holink}_\pi(X, A \times \Delta^k)$ be the fibre deformation explicitly defined in 5.2. Thus, there is a fibre homotopy inverse $j : \mathrm{holink}_\pi(X, A \times \Delta^k) \to \mathrm{holink}_\pi(Y', A \times \Delta^k)$ for i_* defined by $j = R_1$. From the definition of R, we have $R(\omega, t)(u) = \omega(s)$ for some s. Define $p : X \to (0, +\infty]$ by $p(x) = 1/d(x, A)$. Define $f : Y \to \overset{\circ}{\mathrm{cyl}}(q)$ by

$$f(x) = \begin{cases} (\hat{H}(x), p(x)) \in \mathrm{holink}_\pi(X, A \times \Delta^k) \times (0, +\infty), & \text{if } x \in Y \setminus (A \times \Delta^k) \\ x, & \text{if } x \in A \times \Delta^k. \end{cases}$$

Let $p_{Y'} : \mathrm{holink}_\pi(Y', A \times \Delta^k) \to Y'$ and $p_Y^+ : \mathrm{holink}_\pi(Y, A \times \Delta^k) \times \mathbb{R} \to Y$ be the evaluations $p_{Y'}(\omega) = \omega(1)$ and

$$p_Y^+(\omega, t) = \begin{cases} \omega(1), & \text{if } t \leq 0 \\ \omega(1/(1+t)), & \text{if } t \geq 0. \end{cases}$$

Define $g : \overset{\circ}{\mathrm{cyl}}(q) \to Y$ so that on $\mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \subseteq \overset{\circ}{\mathrm{cyl}}(q)$, g is the composition

$$\mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \xrightarrow{j \times \mathrm{id}_\mathbb{R}} \mathrm{holink}_\pi(Y', A \times \Delta^k) \times \mathbb{R} \xrightarrow{p_{Y'} \times \mathrm{id}_\mathbb{R}}$$

$$Y' \times \mathbb{R} \xrightarrow{\hat{H} \times \mathrm{id}_\mathbb{R}} \mathrm{holink}_\pi(Y, A \times \Delta^k) \times \mathbb{R} \xrightarrow{p_Y^+} Y$$

and on $A \times \Delta^k \subseteq \overset{\circ}{\mathrm{cyl}}(q)$, g is the identity. Define the homotopy $F : Y \times I \to X$ by

$$F(x, t) = \begin{cases} (\hat{H}[(R_{1-t}(\hat{H}(x)))(1)])(\frac{d(x,A)+t}{d(x,A)+1}), & \text{if } x \in Y \setminus (A \times \Delta^k) \\ x, & \text{if } x \in A \times \Delta^k. \end{cases}$$

Define

$$\gamma : \mathrm{holink}_\pi(X, A \times \Delta^k) \times (0, 1] \to \mathrm{holink}_\pi(X, A \times \Delta^k)$$

by $\gamma(\omega, t) = \hat{H}[\hat{H}(x_\omega)(t)]$ where $x_\omega = j(\omega)(1) \in Y'$. Define $G' : \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \times I \to \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R}$ by

$$G'(\omega, t, s) = \begin{cases} (\gamma(\omega, \frac{1}{1+t}), (1-s)p[\hat{H}(x_\omega)(\frac{1}{1+t})] + st), & \text{if } s \geq t \\ (\gamma(\omega, 1), (1-s)p[\hat{H}(x_\omega)(1)] + st), & \text{if } s \geq t. \end{cases}$$

Note that $G'_0 = fg| : \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \to \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R}$ and that G' extends via the indentity on $A \times \Delta^k$ to $G' : \overset{\circ}{\mathrm{cyl}}(q) \times I \to \overset{\circ}{\mathrm{cyl}}(q)$. We claim that there exists a homotopy

$$G'' : \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \times I \to \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R}$$

such that

$$G_0''(\omega,t)=\begin{cases}\gamma(\omega,\frac{1}{1+t}), & \text{if } t\geq 0\\ \gamma(\omega,1), & \text{if } t\leq 0.\end{cases}$$

To this end note that by contracting $(0,1]$ to $\{1\}$ there is defined a homotopy $\gamma\simeq\gamma'$ with

$$\gamma'(\omega,t)=\hat{H}[\hat{H}(w_\omega)(1)]=\hat{H}(x_\omega)=\hat{H}(p_{Y'}(j(\omega)).$$

And it is not difficult to see that $\hat{H}p_{Y'}:\mathrm{holink}_\pi(Y',A\times\Delta^k)\to\mathrm{holink}_\pi(Y,A\times\Delta^k)$ is homotopic to the inclusion i_*. Since j is a homotopy inverse for i_*, the homotopy G'' exists as claimed. We can now define the homotopy

$$G:\mathrm{holink}_\pi(X,A\times\Delta^k)\times\mathbb{R}\times I\to\mathrm{holink}_\pi(X,A\times\Delta^k)\times\mathbb{R}$$

by

$$(\omega,t,s)\mapsto\begin{cases}G'(\omega,t,2s), & \text{if } 0\leq s\leq 1/2\\ (G''(\omega,t,2s-1),t), & \text{if } 1/2\leq s\leq 1\end{cases}$$

and extending G via the identity on $A\times\Delta^k$ to get

$$G:\overset{\circ}{\mathrm{cyl}}(q)\times I\to\overset{\circ}{\mathrm{cyl}}(q).$$

For the verification of the properties, see [28, 9.13]. $\square$

Lemma 6.6. *Let $p:E\to B$ be a fibration with B a weakly locally contractible compact metric space. If the fibre of p is finitely dominated, then there exist a compact subspace $K\subseteq E$ and a f.p. homotopy $D:E\times I\to E$ such that $D_0(E)\subseteq K$ and $D_1=\mathrm{id}_E$.*

Proof. Each $x\in B$ has an open neighborhood U_x such that the inclusion $U_x\hookrightarrow B$ is null–homotopic. It follows that there is a fibre homotopy equivalence $f_x: p^{-1}(U_x)\to p^{-1}(x)\times U_x$ over U_x. Let $g_x:p^{-1}(x)\times U_x\to p^{-1}(U_x)$ be a fibre homotopy inverse and $H^x:p^{-1}(U_x)\times I\to p^{-1}(U_x)$ a f.p. homotopy such that $H_0^x=g_xf_x$ and $H_1^x=\mathrm{id}_{p^{-1}(U_x)}$. Since $p^{-1}(x)$ is finitely dominated there exist a compact subspace $K_x\subseteq p^{-1}(x)$ and a homotopy $D^x:p^{-1}(x)\times I\to p^{-1}(x)$ such that $D_0^x(p^{-1}(x))\subseteq K_x$ and $D_1^x=\mathrm{id}_{p^{-1}(x)}$. Let $\hat{D}^x=D^x\times\mathrm{id}_{U_x}:p^{-1}(x)\times U_x\times I\to p^{-1}(x)\times U_x$. Let $\rho_x:B\to I$ be a map such that $\rho_x^{-1}(0)\subseteq U_x$ is a neighborhood of x and $B\setminus U_x\subseteq\rho_x^{-1}(1)$. Define a f.p. homotopy $G^x:E\times I\to E$ by

$$G^x(y,t)=\begin{cases}g_x\hat{D}^x(f_x(y),(1-t)2\rho_x(y)+t), & \text{if } 0\leq\rho_x(y)\leq 1/2\\ H^x(y,t(2\rho_x(y)-1)+(1-t)), & \text{if } 1/2\leq\rho_x(y)\leq 1.\end{cases}$$

Define a f.p. homotopy $F^x:E\times I\to E$ by

$$F^x(y,t)=\begin{cases}H^x(y,t), & \text{if } 0\leq\rho_x(y)\leq 1/2\\ H^x(y,(1-t)(2\rho_x(y)-1)+t), & \text{if } 1/2\leq\rho_x(y)\leq 1.\end{cases}$$

Then $F_0^x=G_1^x$ and $F_1^x=\mathrm{id}_E$. Define a f.p. homotopy $\tilde{D}^x:E\times I\to E$ by

$$\tilde{D}^x(y,t)=\begin{cases}G^x(y,2t), & \text{if } 0\leq t\leq 1/2\\ F^x(y,2t-1), & \text{if } 1/2\leq t\leq 1.\end{cases}$$

Then $\tilde{D}^x_0 = G^x_0$ and $\tilde{D}^x_1 = \mathrm{id}_E$. The compact subspace $C_x = g_x(K_x \times p^{-1}(\rho_x^{-1}(0)))$ of E is such that $\tilde{D}^x_0(\rho_x^{-1}(0)) \subseteq C_x$. Let $\{x_1, \dots, x_k\}$ be a finite subset of B such that $B = \cup_{i=1}^k \rho_{x_i}^{-1}(0)$. Define $D : E \times I \to E$ by

$$D_t = \tilde{D}^{x_k}_t \circ \cdots \circ \tilde{D}^{x_1}_t.$$

Then $D_1 = \mathrm{id}_E$ and

$$D_0(E) \subseteq [\tilde{D}^{x_k}_0 \circ \cdots \circ \tilde{D}^{x_2}_0(C_{x_1})] \cup [\tilde{D}^{x_k}_0 \circ \cdots \circ \tilde{D}^{x_3}_0(C_{x_2})] \cup \cdots \cup [\tilde{D}^{x_k}_0(C_{x_{k-1}})] \cup [C_{x_k}]$$

which is compact as required. $\square$

Proposition 6.7. *Suppose X is a locally compact separable metric space, A is weakly locally contractible, compact space, $A \times \Delta^k$ is sliced forward tame in X with respect to π, and $(X, A \times \Delta^k)$ has finitely dominated local holinks. For every open cover $\mathcal{U}$ of $A \times \mathbb{R} \times \Delta^k$, there exists a strong f.p. $\mathcal{U}$–homotopy equivalence near $A \times \Delta^k$ $(\bar{f}, \bar{g}, X_1', X_2') : X \to \overset{\circ}{\mathrm{cyl}}(q)$.*

Proof. (cf. [28, 9.14].) Let Y, f, g, F, G be as in Proposition 6.5. By Lemma 6.6 there exist a compact subspace $K \subseteq \mathrm{holink}_\pi(X, A \times \Delta^k)$ and a f.p. homotopy $D : \mathrm{holink}_\pi(X, A \times \Delta^k) \times I \to \mathrm{holink}_\pi(X, A \times \Delta^k)$ such that $D_0(\mathrm{holink}_\pi(X, A \times \Delta^k)) \subseteq K$ and $D_1 = \mathrm{id}$. Define $\hat{D} : \overset{\circ}{\mathrm{cyl}}(q) \times I \to \overset{\circ}{\mathrm{cyl}}(q)$ by

$$\hat{D}_s = \begin{cases} D_s \times \mathrm{id}_{\mathbb{R}}, & \text{on } \mathrm{holink}_\pi(X, A \times \Delta^k) \times \mathbb{R} \\ \mathrm{id}, & \text{on } A \times \Delta^k. \end{cases}$$

Define $g' : \overset{\circ}{\mathrm{cyl}}(q) \to Y$ by $g' = g\hat{D}_0$. Define $F' : Y \times I \to X$ by

$$F'_s = \begin{cases} ig\hat{D}_{2s}f, & \text{if } 0 \le s \le 1/2 \\ F_{2s-1}, & \text{if } 1/2 \le s \le 1. \end{cases}$$

Note that $F' : ig'f \simeq i$. Define $G' : \overset{\circ}{\mathrm{cyl}}(q) \times I \to \overset{\circ}{\mathrm{cyl}}(q)$ by

$$G'_s = \begin{cases} G_{2s}\hat{D}_0, & \text{if } 0 \le s \le 1/2 \\ \hat{D}_{2s-1}, & \text{if } 1/2 \le s \le 1. \end{cases}$$

Note that $G' : fg' \simeq \mathrm{id}$. As in [28, 9.14] it is possible to choose a homeomorphism $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma = \mathrm{id}$ on $(-\infty, 0]$ inducing a homeomorphism $\bar{\gamma} : \overset{\circ}{\mathrm{cyl}}(q) \to \overset{\circ}{\mathrm{cyl}}(q)$ such that $\bar{f} = \bar{\gamma}f$ is the desired equivalence with inverse $\bar{g} = \bar{\gamma}^{-1}g'$. ($Q$ plays the role of p in Definition 6.1.) $\square$

Theorem 6.8. *Suppose X is a locally compact separable metric space, $(X, A \times \Delta^k)$ is a sliced homotopically stratified pair with finitely dominated local holinks, A is a compact ANR and $p : X \to A \times (-\infty, +\infty] \times \Delta^k$ is a f.p. proper map with $p| : A \times \Delta^k = p^{-1}(A \times \{+\infty\} \times \Delta^k) \to A \times \{+\infty\} \times \Delta^k$ the identity. Then for every open cover $\mathcal{U}$ of $A \times \mathbb{R} \times \Delta^k$, there exist a compact neighborhood N of $A \times \Delta^k$ in X and a f.p. strict homotopy $p|N \simeq p' : N \to A \times (-\infty, +\infty] \times \Delta^k$* rel *$A \times \Delta^k$ such*

that p' is a sliced $\mathcal{U}$–fibration over $A\times(0,+\infty)\times\Delta^k$ and $(p')^{-1}(A\times(0,+\infty)\times\Delta^k)$ is open in X.

Proof. Given the open cover $\mathcal{U}$ choose an open cover $\mathcal{V}$ such that $\mathrm{st}^2(\mathcal{V})$ refines $\mathcal{U}$. According to Proposition 6.7 there exists a strong f.p. $\mathcal{V}$–homotopy equivalence near $A\times\Delta^k$ $(\bar f,\bar g,X_1',X_2'):X\to\overset{\circ}{\mathrm{cyl}}(q)$ such that X_1' is compact. Let $p''=Q\bar f:X_1'\to A\times(-\infty,+\infty]\times\Delta^k$. Since $(X,A\times\Delta^k)$ is sliced forward tame there exist a compact neighborhood N of $A\times\Delta^k$ in X and a f.p. nearly strict deformation r of N into $A\times\Delta^k$ with $N\subseteq X_1'$ and $r:N\times I\to X_1'$. We show that there exists a f.p. strict homotopy $H:p|N\simeq p''|N$ rel $A\times\Delta^k$ as follows. Let $\pi_1:A\times(-\infty,+\infty]\times\Delta^k\to A\times\Delta^k$ and $\pi_2:A\times(-\infty,+\infty]\times\Delta^k\to(-\infty,+\infty]$ denote the projections. Define $H:N\times I\to A\times(-\infty,+\infty]\times\Delta^k$ by

$$\pi_1H(x,t)=\begin{cases}pr(x,2t), & \text{if } 0\le t\le 1/2\\ p''r(x,2-2t), & \text{if } 1/2\le t\le 1\end{cases}$$

and $\pi_2H(x,t)=(1-t)\pi_2p(x)+t\pi_2p''(x)$. According to Proposition 6.3 there exists an $m>0$ such that p'' is a sliced $\mathcal{U}$–fibration over $(A\times(m,+\infty)\times\Delta^k)$. We may assume that $(p'')^{-1}(A\times(m,+\infty)\times\Delta^k)\subseteq N$. We conclude the proof by defining an isotopy $G:A\times(-\infty,+\infty]\times\Delta^k\times I\to A\times(-\infty,+\infty]\times\Delta^k$ by $G(x,s,t,u)=(x,s-um,t)$ and setting $p'=G_1p''$. Since $G_0=\mathrm{id}$, $A\times(0,+\infty]\times\Delta^k=G_1(A\times(m,+\infty)\times\Delta^k)$ and G_1 is an isometry, it follows that $G_up'':p''\simeq p'$, $0\le u\le 1$, and p' is the desired map. $\square$

7. Higher classification of stratified neighborhoods

Throughout this section B will denote a fixed closed manifold. We will prove Theorem 2.3, the main result of this paper, which classifies families of neighborhoods of B in stratified pairs with B as the lower stratum. This higher classification is given in terms of families of manifold approximate fibrations over $B\times\mathbb{R}$. In fact, Theorem 2.3 asserts that the teardrop construction defines a homotopy equivalence between the moduli space of manifold approximate fibrations over $B\times\mathbb{R}$ and the moduli space of stratified neighborhoods of B. There are two aspects of the proof: existence and uniqueness. Existence essentially means that the simplicial map between moduli spaces is surjective on homotopy groups, whereas uniqueness means that the map is injective on homotopy groups. The actual proof combines both aspects by verifying that the map is 'relatively surjective' on homotopy groups. However, the two aspects are evident in the lead-up to the proof.

The existence problem involves showing that a family (parametrized by Δ^k) of stratified neighborhoods of B is given by the teardrop of a family of manifold approximate fibrations over $B\times\mathbb{R}$. The precise statement is Proposition 7.2. It is proved by first appealing to Theorem 6.8 which establishes that such a family of neighborhoods is given by the teardrop of a family of $\mathcal{U}$–fibrations over $B\times\mathbb{R}$ where $\mathcal{U}$ is an arbitrarily small open cover of $B\times\mathbb{R}$. Then we use sucking phenomena for manifold approximate fibrations, which says that if $\mathcal{U}$ is sufficiently fine then a $\mathcal{U}$–fibration deforms to a manifold approximate fibration. Sucking phenomena for approximate fibrations were first discovered by Chapman [7], [8], but the family version which we require appears in [24]. The technical version of sucking which we require is stated in Proposition 7.1. We point out below that Proposition 7.2

together with the material from §4 suffices to give a proof of Theorem 2.1 (Teardrop Neighborhood Existence) even though it also follows from Theorem 2.3.

Just as the existence aspect is based on a fundamental phenomenon of manifold approximate fibrations, the uniqueness aspect is based on another such phenomenon of manifold approximate fibrations: two families of close manifold approximate fibrations can be connected by a close family of manifold approximate fibrations (parametrized by Δ^k). In other words, the moduli space of manifold approximate fibrations is locally k–connected for each $k \geq 0$. This phenomenon was observed in [24]. Lemma 7.3 contains an elementary argument which shows how we get into a situation of having two close families of manifold approximate fibrations. Proposition 7.4 is the technical version of the local connectivity result which we require and Proposition 7.5 sets the stage for how it is used in the proof of the classification theorem.

We begin by quoting the version of the sucking phenomena which we will use.

Proposition 7.1 (Sucking). *Let $n \geq 5$ and $k \geq 0$. For every open cover $\mathcal{U}$ of $B \times \mathbb{R} \times \Delta^k$ there exists an open cover $\mathcal{V}$ of $B \times \mathbb{R} \times \Delta^k$ such that if M is an n–manifold (without boundary), $N \subseteq M \times \Delta^k$ is a closed subset, $j : N \to B \times \mathbb{R} \times \Delta^k$ is a f.p. proper map such that j is a sliced $\mathcal{V}$–fibration over $B \times (0, +\infty) \times \Delta^k$, and $j^{-1}(B \times (0, +\infty) \times \Delta^k)$ is an open subspace of $M \times \Delta^k$, then j is f.p. properly $\mathcal{U}$–homotopic rel $j^{-1}(B \times (-\infty, 0] \times \Delta^k)$ to a map $j' : N \to B \times \mathbb{R} \times \Delta^k$ with j' a sliced approximate fibration over $B \times (1, +\infty) \times \Delta^k$.*

Proof. See [24], [29, §13]. □

In the next result we combine the homotopy information of the previous section (Theorem 6.8) with the sucking result (Proposition 7.1) to prove the existence of manifold approximate fibration teardrop structure for manifold stratified neighborhoods.

Proposition 7.2. *If $n \geq 5$ and $\pi : (X, B \times \Delta^k) \to \Delta^k$ is a k–simplex of $\mathrm{SN}^n(B)$, then there exists a compact neighborhood $\widehat{N}$ of $B \times \Delta^k$ in X and a f.p. proper strict map*

$$\widehat{p} : (\widehat{N}, B \times \Delta^k) \to (B \times (-\infty, +\infty] \times \Delta^k, B \times \{+\infty\} \times \Delta^k) \qquad \text{rel } B \times \Delta^k$$

such that $\widehat{p}$ is a sliced approximate fibration over $B \times (1, +\infty) \times \Delta^k$.

Proof. Choose an open cover $\mathcal{U}$ of $B \times \mathbb{R} \times \Delta^k$ such that

$$\mathrm{lub}\{\mathrm{diam}(U) \mid U \in \mathcal{U}, U \cap (B \times [m, +\infty) \times \Delta^k \neq \emptyset\} \to 0 \text{ as } m \to \infty.$$

Let $\mathcal{V}$ be an open cover of $B \times \mathbb{R} \times \Delta^k$ given by Proposition 7.1 which depends on $\mathcal{U}$. Since $B \times \Delta^k$ is sliced forward tame in X, it follows that there exist a compact neighborhood N_0 of $B \times \Delta^k$ in X and a f.p. retraction $r : N_0 \to B \times \Delta^k$. We may assume that N_0 is contained in a trivial neighborhood of $B \times \Delta^k$ (in the sense of Definition 5.1). Let $N = \mathrm{int}(N_0)$ and choose a proper map $u : N \to (-\infty, +\infty]$ such that $u^{-1}(+\infty) = B \times \Delta^k$. Define $p' : N \to B \times (-\infty, +\infty] \times \Delta^k$ by $p'(x) = (\mathrm{proj}_B r(x), u(x), \mathrm{proj}_{\Delta^k} r(x))$. Note that p' is a f.p. proper strict map and rel $B \times \Delta^k$. Since $(X, B \times \Delta^k)$ is a sliced manifold stratified pair, so is $(N, B \times \Delta^k)$

(Proposition 5.2). Theorem 6.8 implies that there exist a compact neighborhood $\widehat{N}$ of $B \times \Delta^k$ in N and a f.p. proper strict homotopy

$$p'|\widehat{N} \simeq p'' : \widehat{N} \to B \times (-\infty, +\infty] \times \Delta^k \qquad \mathrm{rel}\ B \times \Delta^k$$

such that p'' is a sliced $\mathcal{V}$–fibration over $B \times (0, +\infty) \times \Delta^k$ and $(p'')^{-1}(B \times (0, +\infty) \times \Delta^k)$ is open in N (and hence open in X). Now Proposition 7.1 and the choice of $\mathcal{V}$ imply that there exists a f.p. proper $\mathcal{U}$–homotopy

$$p''|\widehat{N} \setminus (B \times \Delta^k) \simeq p''' : \widehat{N} \setminus (B \times \Delta^k) \to B \times \mathbb{R} \times \Delta^k$$

such that p''' is a sliced approximate fibration over $B \times (1, +\infty) \times \Delta^k$. (We are in a product situation as required by Proposition 7.1 because N_0 was chosen to be in a trivial neighborhood.) The defining property of the open cover $\mathcal{U}$ implies that the map p''' extends via the identity on $B \times \Delta^k$ to a map

$$\widehat{p} : \widehat{N} \to B \times (-\infty, +\infty] \times \Delta^k. \quad \square$$

As mentioned in §2 we can now give a proof of Theorem 2.1 (on the existence of teardrop neighborhoods) which avoids some of the machinery required for the proof of Theorem 2.3.

Proof of Theorem 2.1 (*Teardrop Neighborhood Existence*). If (X, B) is a manifold stratified pair with $\dim(X \setminus B) = n \geq 5$, then (X, B) is a vertex of $\mathrm{SN}^n(B)$. It follows from Proposition 7.2 that B has a neighborhood in X which is the teardrop of a manifold approximate fibration. The converse follows from Corollary 4.11. $\square$

We are now ready to begin the uniqueness aspects of the main result. The first lemma shows how to modify two teardrop collapse maps so that they become close near the lower stratum.

Lemma 7.3. *Suppose B, K are compact metric spaces, X is a locally compact metric space containing $B \times K$ with a map $\pi : X \to K$ such that $\pi| : B \times K \to K$ is projection. Suppose $p, q : (X, B \times K) \to (B \times (-\infty, +\infty] \times K, B \times \{+\infty\} \times K)$ are two fibre preserving* (*with respect to π*) *strict maps which are the identity on $B \times K$ and proper over $B \times (0, +\infty) \times K$. For every open cover $\mathcal{V}$ of $B \times \mathbb{R} \times K$ there exists a f.p. strict isotopy $H : B \times (-\infty, +\infty] \times K \times I \to B \times (-\infty, +\infty] \times K \times I$ rel $(B \times (-\infty, 0] \times K) \cup (B \times \{+\infty\} \times K)$ such that $p' = H_1 p$ and $q' = H_1 q$ are $\mathcal{V}$–close over $B \times (1, +\infty) \times K$* (*meaning if $x \in (p')^{-1}(B \times (1, +\infty) \times K) \cup (q')^{-1}(B \times (1, +\infty) \times K)$, then there exists $V \in \mathcal{V}$ such that $p'(x), q'(x) \in V$*).

Proof. Assume $B \times K$ has a fixed metric, $\mathbb{R}$ has the standard metric and $B \times \mathbb{R} \times K$ has the product metric. For each $n = -1, 0, 1, 2, \ldots$ let $\epsilon_n > 0$ be a Lebesque number for the open cover $\{V \cap (B \times [n, n+1] \times K) \mid V \in \mathcal{V}\}$ of $B \times [n, n+1] \times K$. We may assume that $\epsilon_{-1} < \epsilon_0 < \epsilon_1 < \ldots$. Using the properness of p, q (over $B \times (0, +\infty) \times K$) and the fact that p, q are the identity on $B \times K$, construct (by induction) a sequence $0 < t_{-1} < t_0 < t_1 < \ldots$ such that $t_n \to \infty$ as $n \to \infty$, p, q are $(\epsilon_n/3)$–close over $B \times [t_n, +\infty] \times K$, and if $x \in p^{-1}(B \times [t_n, t_{n+1}] \times K) \cup q^{-1}(B \times [t_n, t_{n+1}] \times K)$, then $p(x), q(x) \in B \times [t_{n-1}, t_{n+2}] \times K$ for each $n = 0, 1, 2, \ldots$. Also construct a sequence $0 = y_0 < y_1 < y_2 < \ldots$ refining $\{0, 1, 2, \ldots\}$ such that $y_n \geq n$ and if $n \leq y_k \leq n+1$, then $y_{k+1} - y_k < \epsilon_{n+1}/3$. Define a homeomorphism

$h' : (-\infty, +\infty] \to (-\infty, +\infty]$ so that for each $n = 0, 1, 2, \ldots$ $h'(t_n) = y_n$, h' is linear on $[t_n, t_{n+1}]$ and is the identity on $(-\infty, 0]$. Define $h = \mathrm{id}_B \times h' \times \mathrm{id}_K : B \times (-\infty, +\infty] \times K \to B \times (-\infty, +\infty] \times K$. The natural isotopy $\mathrm{id}_{(-\infty,+\infty]} \simeq h'$ induces an isotopy $H : \mathrm{id}_{B\times(-\infty,+\infty]\times K} \simeq h = H_1$ and one checks that $p' = H_1 p$ and $q' = H_1 q$ satisfy the conclusions. $\square$

The next result formulates the version of local connectivity for families of manifold approximate fibrations which we require. Then Proposition 7.5 applies it in the situation which will arise in the proof of the main result.

Proposition 7.4. *Suppose that $n \geq 5$ and K is a compact polyhedron. For every open cover $\mathcal{U}$ of $B \times \mathbb{R} \times K$ there exists an open cover $\mathcal{V}$ of $B \times \mathbb{R} \times K$ such that if $\pi : M \to K$ is a fibre bundle projection with n–manifold fibres (without boundary), $N \subseteq M$ is a closed subset, $p_1, p_2 : N \to B \times \mathbb{R} \times K$ are two f.p. proper maps which are $\mathcal{V}$–close over $B \times (0, +\infty) \times K$ and sliced approximate fibrations over $B \times (0, +\infty) \times K$, and $p_i^{-1}(B \times (0, +\infty) \times K)$ is open in M for $i = 1, 2$, then there exists a f.p. proper $\mathcal{U}$–homotopy $F : p_1 \simeq p_2$ such that $F_s : N \to B \times \mathbb{R} \times K$ is a sliced approximate fibration over $B \times (1, +\infty) \times K$ for each $0 \leq s \leq 1$.*

Proof. This just involves minor modifications in the arguments of [24] used to prove that spaces of manifold approximate fibrations are locally k–connected for each $k \geq 0$. $\square$

Proposition 7.5. *Suppose K is a compact polyhedron and $\pi : (Y, B \times K) \to B \times K$ is a sliced manifold stratified pair with $\dim \pi^{-1}(u) = n \geq 5$ for $u \in K$ for which there is a f.p. proper strict map*

$$p : (Y, B \times K) \to (B \times (-\infty, +\infty] \times K, B \times \{+\infty\} \times K) \text{ rel } B \times K$$

which is a sliced manifold approximate fibration over $B \times \mathbb{R} \times K$. Suppose $t \in \mathbb{R}$ and $\widehat{Y}$ is an open neighborhood of $B \times K$ in Y for which there is a f.p. proper strict map

$$\widehat{p} : (\widehat{Y}, B \times K) \to (B \times (t, +\infty] \times K, B \times \{+\infty\} \times K) \text{ rel } B \times K$$

which is a sliced manifold approximate fibration over $B \times (t, +\infty) \times K$. Then there exist $t_2 > t$, a compact neighborhood X of $B \times K$ in Y with $X \subseteq \widehat{Y}$ and a f.p. strict homotopy

$$F : p|X \simeq \widehat{p}|X : X \to B \times (-\infty, +\infty] \times K \text{ rel } B \times K$$

which is proper over $B \times (t_2, +\infty] \times K$ and such that $F_s : X \to B \times (-\infty, +\infty] \times K$ is a sliced manifold approximate fibration over $B \times (t_2, +\infty) \times K$ for each $0 \leq s \leq 1$.

Proof. Choose an open cover $\mathcal{U}$ of $B \times \mathbb{R} \times K$ such that

$$\mathrm{lub}\{\mathrm{diam}(U) \mid U \in \mathcal{U}, U \cap (B \times [m, +\infty) \times \Delta^k \neq \emptyset\} \to 0 \text{ as } m \to \infty.$$

Let $\mathcal{V}$ be the open cover of $B \times \mathbb{R} \times K$ given by Proposition 7.4 which depends on $\mathcal{U}$. Let W be a locally trivial neighborhood of $B \times K$ in Y (in the sense of Definition 5.1) and assume that $W \subseteq \widehat{Y}$. Choose $t_0 \geq t$ such that

$$p^{-1}(B \times [t_0, +\infty] \times K) \cup \widehat{p}^{-1}(B \times [t_0, +\infty] \times K) \subseteq W.$$

Let
$$X = p^{-1}(B \times [t_0, +\infty] \times K) \cap \widehat{p}^{-1}(B \times [t_0, +\infty] \times K).$$
Choose $t_1 > t_0$ such that
$$p^{-1}(B \times [t_1, +\infty] \times K) \cup \widehat{p}^{-1}(B \times (t_1, +\infty] \times K) \subseteq X$$
and note that $p|, \widehat{p}| : X \to B \times (t_0, +\infty] \times K$ are proper over $B \times (t_1, +\infty] \times K$ and sliced approximate fibrations over $B \times (t_1, +\infty) \times K$. Let $t_2 = t_1 + 1$. Lemma 7.3 can be applied to yield a f.p. strict isotopy
$$H : B \times (-\infty, +\infty] \times K \times I \to B \times (-\infty, +\infty] \times K \times I$$
$$\text{rel } (B \times (-\infty, t_1] \times K) \cup (B \times \{+\infty\} \times K)$$
such that $p' = H_1 p|X$ and $q' = H_1 \widehat{p}|X$ are $\mathcal{V}$–close over $B \times (t_2, +\infty] \times K$. Because H is rel $B \times (-\infty, t_1] \times K$, p' and q' are sliced approximate fibrations over $B \times (t_1, +\infty] \times K$. Proposition 7.4 can be applied to yield a f.p. $\mathcal{U}$–homotopy $F : p'| \simeq q'| : X \setminus (B \times K) \to B \times \mathbb{R} \times K$ such that $F_s : X \setminus (B \times K) \to B \times \mathbb{R} \times K$ is a sliced approximate fibration over $B \times (t_2, +\infty) \times K$ for each $0 \leq s \leq 1$. The choice of the open cover $\mathcal{U}$ implies that F extends via the identity $B \times K \to B \times \{+\infty\} \times K$ to a homotopy (also denoted F) $F : p' \simeq q' : X \to B \times (-\infty, +\infty] \times K$. □

We need one more lemma before proving the main result.

Lemma 7.6. *If $n \geq 5$ and $t \in \mathbb{R}$, then the restriction $\rho : \mathrm{MAF}^n(B \times \mathbb{R}) \to \mathrm{MAF}^n(B \times (t, +\infty))$ is a homotopy equivalence.*

Proof. First observe that the techniques of [29, §3] show that ρ is in fact a simplicial map. There are a couple of approaches to proving that ρ is a homotopy equivalence. One is to use geometric techniques as presented in [29, §14] in proving uniqueness of fibre germs. The other is to use the Manifold Approximate Fibration Classification Theorem [29], [30] and observe that restriction induces a homotopy equivalence of the classifying spaces. □

Let $n \geq 5$. We prove the main theorem by showing that $\Psi : \mathrm{MAF}^n(B \times \mathbb{R}) \to \mathrm{SN}^n(B)$ (as constructed in §5) is a homotopy equivalence. Since both these simplicial sets satisfy the Kan condition, it suffices to show that Ψ induces an isomorphism on homotopy groups (including π_0). To accomplish this suppose that we are given the following set-up.

Data 7.7. Suppose $k \geq 0$.

(1) Let $\pi : (X, B \times \Delta^k) \to \Delta^k$ be a k-simplex of $\mathrm{SN}^n(B)$.
(2) Let $p : M \to B \times \mathbb{R} \times \partial\Delta^k$ be a union of $(k+1)$ $(k-1)$-simplices of $\mathrm{MAF}^n(B \times \mathbb{R})$.
(3) Suppose for each $i = 0, \cdots k$, the $(k-1)$-simplex $\pi| : (\pi^{-1}(\partial_i\Delta^k), B \times \partial_i\Delta^k) \to \partial_i\Delta^k$ of $\mathrm{SN}^n(B)$ is the image under Ψ of the $(k-1)$-simplex $p| : p^{-1}(B \times \mathbb{R} \times \partial_i\Delta^k) \to B \times \mathbb{R} \times \partial_i\Delta^k$ of $\mathrm{MAF}^n(B \times \mathbb{R})$ so that $M = \pi^{-1}(\partial\Delta^k) \setminus (B \times \partial\Delta^k)$.

Note that if $k = 0$, then only item (1) is meaningful.

Theorem 7.8. *Given Data 7.7, there is a k-simplex $\widetilde{p}:\widetilde{M}\to B\times\mathbb{R}\times\Delta^k$ of* $\mathrm{MAF}^n(B\times\mathbb{R})$ *which equals p over $B\times\mathbb{R}\times\partial\Delta^k$ and whose image under Ψ is homotopic in* $\mathrm{SN}^n(B)$ *to π* rel ∂. *Hence, $\Psi:\mathrm{MAF}^n(B\times\mathbb{R})\to\mathrm{SN}^n(B)$ induces an isomorphism on homotopy groups and is a homotopy equivalence.*

Proof. According to Proposition 7.2, there exists a compact neighborhood $\widehat{N}$ of $B\times\Delta^k$ in X and a f.p. proper strict map

$$\widehat{p}:(\widehat{N},B\times\Delta^k)\to(B\times(-\infty,+\infty]\times\Delta^k,B\times\{+\infty\}\times\Delta^k)\qquad\mathrm{rel}\ B\times\Delta^k$$

such that $\widehat{p}$ is a sliced approximate fibration over $B\times(1,+\infty)\times\Delta^k$. Choose $t\geq 1$ such that $\widehat{p}^{-1}(B\times(t,+\infty]\times\Delta^k)$ is open in X. Let $Y=\partial X=\pi^{-1}(\partial\Delta^k)$ which by assumption is the teardrop $M\cup_p(B\times\partial\Delta^k)$. Extend $p:M\to B\times\mathbb{R}\times\partial\Delta^k$ via the identity $B\times\partial\Delta^k\to B\times\{+\infty\}\times\partial\Delta^k$ to $p_+:Y\to B\times B\times(-\infty,+\infty]\times\partial\Delta^k$ which is continuous since it is the teardrop collapse. Let $\widehat{Y}=\widehat{p}^{-1}(B\times(t,+\infty]\times\partial\Delta^k)$. Since $\widehat{Y}$ is open in Y, it follows that $\widehat{p}|:\widehat{Y}\to B\times(t,+\infty]\times\partial\Delta^k$ is a sliced manifold approximate fibration over $B\times(t,+\infty)\times\partial\Delta^k$. It follows from Proposition 7.5 applied with $K=\partial\Delta^k$ that there exist $t_2>t$, a compact neighborhood $\widetilde{Y}$ of $B\times\partial\Delta^k$ in Y with $\widetilde{Y}\subseteq\widehat{Y}$, and a f.p. strict homotopy

$$F:p_+|\widetilde{Y}\simeq\widehat{p}|\widetilde{Y}:\widetilde{Y}\to B\times(-\infty,+\infty]\times\partial\Delta^k$$

which is proper over $B\times(t_2,+\infty]\times\partial\Delta^k$ and such that $F_s:\widetilde{Y}\to B\times(-\infty,+\infty]\times\partial\Delta^k$ is a sliced manifold approximate fibration over $B\times(t_2,+\infty)\times\partial\Delta^k$ for each $0\leq s\leq 1$. Consider F as a map $F:\widetilde{Y}\times I\to B\times(-\infty,+\infty]\times\partial\Delta^k\times I$. Choose $t_3\geq t_2$ such that $F^{-1}(B\times(t_3,+\infty]\times\partial\Delta^k\times I)$ is open in $Y\times I$ and let $W=F^{-1}(B\times(t_3,+\infty)\times\partial\Delta^k\times I)$. Since the composition

$$W\xrightarrow{F}B\times(t_3,+\infty)\xrightarrow{\mathrm{proj}}\partial\Delta^k\times I$$

is a submersion and $F|:W\to B\times(t_3,+\infty)\times\partial\Delta^k\times I$ is a sliced (over $\partial\Delta^k\times I$) manifold approximate fibration, it follows from [25, Lemma 4.1] that $W\to\partial\Delta^k\times I$ is a fibre bundle projection. Let $W_0=p^{-1}(B\times(t_3,+\infty)\times\partial\Delta^k)$ and $W_1=\widehat{p}^{-1}(B\times(t_3,+\infty)\times\partial\Delta^k)$. It follows that $F|W$ may be thought of as a homotopy in $\mathrm{MAF}(B\times(t_3,+\infty))$ from $p|:W_0\to B\times(t_3,+\infty)\times\partial\Delta^k$ to $\widehat{p}|:W_1\to B\times(t_3,+\infty)\times\partial\Delta^k$.

Now consider the open subspace $\widehat{X}=\widehat{p}^{-1}(B\times(t_3,+\infty]\times\Delta^k)$ of X and let $\widehat{M}=\widehat{X}\setminus(B\times\Delta^k)=\widehat{p}^{-1}(B\times(t_3,+\infty)\times\Delta^k)$. Since $\widehat{p}|:\widehat{X}\to B\times(t_3,+\infty]\times\Delta^k$ is a sliced manifold approximate fibration over $B\times(t_3,+\infty)\times\Delta^k$, it follows using [25, Lemma 4.1] again that $\widehat{p}:\widehat{M}\to B\times(t_3,+\infty)\times\Delta^k$ is a k–simplex of $\mathrm{MAF}(B\times(t_3,+\infty))$. Its boundary is $\widehat{p}|=F_1|:W_1\to B\times(t_3,+\infty)\times\partial\Delta^k$.

Let $\rho:\mathrm{MAF}(B\times\mathbb{R})\to\mathrm{MAF}(B\times(t_3,+\infty))$ be the simplicial map induced by restriction. It is a homotopy equivalence by Lemma 7.6. Define a simplicial map $\Psi':\mathrm{MAF}(B\times(t_3,+\infty))\to\mathrm{SN}(B)$ induced by the teardrop construction in analogy to the map $\Psi:\mathrm{MAF}(B\times\mathbb{R})\to\mathrm{SN}(B)$. In fact, if $q:Q\to B\times\mathbb{R}\times\Delta^k$ is a k–simplex of $\mathrm{MAF}(B\times\mathbb{R})$, then $\Psi'\rho(q)=q^{-1}(B\times(t_3,+\infty)\times\Delta^k)\cup_q(B\times\Delta^k)$ is an open subspace of $\Psi(q)=Q\cup_q(B\times\Delta^k)$ and the mapping cylinder of the inclusion induces a homotopy in $\mathrm{SN}(B)$ from $\Psi'\rho(q)$ to $\Psi(q)$ (see §5). In this way we construct a homotopy

$$\mathcal{CYL}:\Psi'\rho\simeq\Psi:\mathrm{MAF}(B\times\mathbb{R})\to\mathrm{SN}(B).$$

Use the homotopy $F|W$ and a collar of $\partial\Delta^k$ in Δ^k to enlarge the k–simplex $\widehat{p}| : \widehat{M} \to B \times (t_3, +\infty) \times \Delta^k$ of $\mathrm{MAF}(B \times (t_3, +\infty))$ to a k–simplex $p^* : M^* \to B \times (t_3, +\infty) \times \Delta^k$ of $\mathrm{MAF}(B \times (t_3, +\infty))$ so that ∂p^* is $\rho(p)$. Note that F is a homotopy in $\mathrm{MAF}(B \times (t_3, +\infty))$ from $\rho(p) = F_0|W_0$ to $\partial(\widehat{p}|\widehat{M}) = F_1|W_1$. Note that since $\Psi'(\widehat{p}|\widehat{M})$ is an open subspace of X, the mapping cylinder construction induces a homotopy $\mathcal{CYL} : \Psi'(\widehat{p}|\widehat{M}) \simeq X$ in $\mathrm{SN}(B)$. Note also that since each $F^{-1}(B \times (t_3, +\infty] \times \partial\Delta^k \times \{s\})$ is an open subspace op ∂X, the mapping cylinder construction induces an extension of the homotopy $\mathcal{CYL} : \Psi'(\widehat{p}|\widehat{M}) \simeq X$ to a homotopy $\mathcal{CYL} : \Psi'(p^*) \simeq X$.

The situation now is that we have a k–simplex p^* of $\mathrm{MAF}(B \times (t_3, +\infty))$ such that $\rho(p) = \partial p^*$ and the mapping cylinder construction induces a homotopy $\mathcal{CYL} : \Psi'(p^*) \simeq X$. Since $\rho : \mathrm{MAF}(B \times \mathbb{R}) \to \mathrm{MAF}(B \times (t_3, +\infty))$ is a homotopy equivalence, there exists a k–simplex $\widetilde{p}$ of $\mathrm{MAF}(B \times \mathbb{R})$ such that $\partial\widetilde{p} = p$ and a homotopy $G : \rho(\widetilde{p}) \simeq p^*$ rel $\partial\rho(\widetilde{p}) = \partial p^*$. Thus $\Psi'(G)$ is a homotopy in $\mathrm{SN}(B)$ from $\psi'\rho(\widetilde{p})$ to $\Psi'(p^*)$ rel ∂. This homotopy taken together with the homotopy $\mathcal{CYL} : \Psi'(p^*) \simeq X$, yields a homotopy $H : \Psi'\rho(\widetilde{p}) \simeq X$ in $\mathrm{SN}(B)$ which restricts to $\mathcal{CYL} : \partial\Psi'\rho(\widetilde{p}) \simeq \partial X$. On the other hand, we have already observed that there is a homotopy $\mathcal{CYL} : \Psi'\rho(\widetilde{p}) \simeq \Psi(\widetilde{p})$. The concatenation $\Psi(\widetilde{p}) \simeq \Psi'\rho(\widetilde{p}) \simeq X$, together with the fact that the two homotopies restrict to inverses on the boundary, implies that there exists a homotopy $\Psi(\widetilde{p}) \simeq X$ rel ∂. $\square$

8. Examples of exotic stratifications

In this section we use the classification of neighborhood germs to construct examples of manifold stratified pairs in which the lower stratum does not have a neighborhood given by the mapping cylinder of a fibre bundle. Moreover, we construct examples in which this phenomenon persists under euclidean stabilization.

Theorem 8.1. *For every integer $m \geq 6$ there exists a locally conelike manifold stratified pair (X, S^1) with $\dim(X \setminus S^1) = m$ such that S^1 has a manifold approximate fibration mapping cylinder neighborhood in X, but for each $i \geq 0$ $S^1 \times \mathbb{R}^i$ does not have a fibre bundle mapping cylinder neighborhood in $X \times \mathbb{R}^i$. In fact, $S^1 \times \mathbb{R}^i$ does not have a block bundle mapping cylinder neighborhood in $X \times \mathbb{R}^i$.*

For the remainder of this section, let F denote a closed connected manifold of dimension n. Let $\mathrm{TOP}^b(F \times \mathbb{R}^i)$ denote the simplicial set of *bounded homeomorphisms* on $F \times \mathbb{R}^i$ so that a k–simplex of $\mathrm{TOP}^b(F \times \mathbb{R}^i)$ consists of a homeomorphism $h : F \times \mathbb{R}^i \times \Delta^k \to F \times \mathbb{R}^i \times \Delta^k$ such that h is fibre preserving over Δ^k and bounded in the $\mathbb{R}^i$–direction. This latter condition means there exists a constant $c > 0$ such that $p_2 h$ is c–close to p_2 where $p_2 : F \times \mathbb{R}^i \times \Delta^k \to \mathbb{R}^i$ is projection.

Let $\mathrm{C}^b(F \times \mathbb{R}^i)$ denote the simplicial set of *bounded concordances* on $F \times \mathbb{R}^i$ so that a k–simplex of $\mathrm{C}^b(F \times \mathbb{R}^i)$ consists of a homeomorphism $h : F \times \mathbb{R}^i \times [0,1] \times \Delta^k \to F \times \mathbb{R}^i \times [0,1] \times \Delta^k$ such that h is fibre preserving over Δ^k, $h| : F \times \mathbb{R}^i \times \{0\} \times \Delta^k \to F \times \mathbb{R}^i \times \{0\} \times \Delta^k$ is the identity, and h is bounded over $\mathbb{R}^i$.

A bounded concordance on $F \times \mathbb{R}^i$ induces a bounded homeomorphism on $F \times \mathbb{R}^i$ by restricting the concordance to $F \times \mathbb{R}^i \times \{1\}$. This defines a simplicial map

$$\rho : \mathrm{C}^b(F \times \mathbb{R}^i) \to \mathrm{TOP}^b(F \times \mathbb{R}^i)$$

by setting $\rho(h) = h| : F \times \mathbb{R}^i \times \{1\} \times \Delta^k = F \times \mathbb{R}^i \times \Delta^k \to F \times \mathbb{R}^i \times \{1\} \times \Delta^k = F \times \mathbb{R}^i \times \Delta^k$.

Euclidean stabilization induces a simplicial map

$$\sigma : \mathrm{TOP}^b(F \times \mathbb{R}^i) \to \mathrm{TOP}^b(F \times \mathbb{R}^{i+1}); \qquad h \mapsto h \times \mathrm{id}_{\mathbb{R}}$$

and, in particular, a group homomorphism $\pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^i) \xrightarrow{\sigma} \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{i+1})$ for each $i \geq 0$.

Proposition 8.2 (Anderson-Hsiang). *There is a homotopy fibration sequence*

$$\mathrm{C}^b(F \times \mathbb{R}^i) \xrightarrow{\rho} \mathrm{TOP}^b(F \times \mathbb{R}^i) \xrightarrow{\sigma} \mathrm{TOP}^b(F \times \mathbb{R}^{i+1}).$$

In particular, there is a short exact sequence

$$\pi_0 \mathrm{C}^b(F \times \mathbb{R}^i) \xrightarrow{\rho} \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^i) \xrightarrow{\sigma} \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{i+1}).$$

Proof. This is essentially the fibration of Anderson-Hsiang [3, 9.3]. One must use [2, Thm. 4] to identify $\mathrm{C}^b(F \times \mathbb{R}^i)$ with the fibre in [3]. Similarly one needs a reinterpretation of $\mathrm{TOP}^b(F \times \mathbb{R}^i)$. See [31, Thm. 1.2] for an explicit proof. See also Lashof-Rothenberg [37, §8]. □

An *inertial* h–cobordism on F is an h–cobordism $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W = F$ and $\partial_1 W$ homeomorphic to F. It is possible to define the simplicial set of h–cobordisms on F (e.g. Waldhausen [59]) and the simplicial set of inertial h–cobordisms on F. However, for this paper we only need the sets of components of these simplicial sets. Thus, let $\pi_0 h\mathrm{cob}(F)$ denote the set of equivalence classes of h–cobordisms on F such that $(W; \partial_0 W, \partial_1 W)$ is equivalent to $(W'; \partial_0 W', \partial_1 W')$ if and only if there exists a homeomorphism $H : W \to W'$ such that $H| : \partial_0 W = F \to \partial_0 W' = F$ is the identity. The set $\pi_0 \mathrm{I}h\mathrm{cob}(F)$ of inertial h–cobordisms on F is the subset of $\pi_0 h\mathrm{cob}(F)$ consisting of all classes represented by inertial h–cobordisms.

The s–cobordism theorem gives a bijection

$$\pi_0 h\mathrm{cob}(F) \xrightarrow{\tau} \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$$

provided $n \geq 5$, which sends an h–cobordism $(W; \partial_0 W, \partial_1 W)$ to the Whitehead torsion $\tau(W, \partial_0 W)$ in $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$. In general, the image of $\pi_0 \mathrm{I}h\mathrm{cob}(F)$ in $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ need not be a subgroup (cf. Hausmann [19], Ling [39]).

We now recall the well-known 'region between' construction (cf. Anderson-Hsiang [2,§8]) which defines a function

$$\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \longrightarrow \pi_0 \mathrm{I}h\mathrm{cob}(F).$$

If $h : F \times \mathbb{R} \to F \times \mathbb{R}$ is a bounded homeomorphism representing a class $[h] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R})$, choose a $L > 0$ so large that $h(F \times \{L\}) \subseteq F \times (0, \infty)$. Let $W = h(F \times (-\infty, L]) \setminus F \times (-\infty, 0)$, $\partial_0 W = F \times \{0\} = F$, and $\partial_1 W = h(F \times \{L\})$. Then $(W; \partial_0 W, \partial_1 W)$ is an inertial h–cobordism on F representing a class $[W] \in \pi_0 \mathrm{I}h\mathrm{cob}(F)$. Set $\beta([h]) = [W]$. The function β is well-defined by the Isotopy Extension Theorem of Edwards-Kirby [11]. One should not confuse $\tau(\beta(h))$ with the torsion of the homotopy equivalence

$$h_1 : F = F \times \{0\} \hookrightarrow F \times \mathbb{R} \xrightarrow{h} F \times \mathbb{R} \xrightarrow{\mathrm{proj}} F.$$

To see the relationship between these two torsions let $j : W \to [0,1]$ be any map with $j^{-1}(0) = \partial_0 W$ and $j^{-1}(1) = \partial_1 W$. Since $F \times \{0\} \hookrightarrow F \times (-\infty, 0]$ and $h(F \times \{L\}) \hookrightarrow h(F \times [L, +\infty))$ are homotopy equivalences, so is the inclusion $i : W \hookrightarrow F \times \mathbb{R}$, and there is a homotopy equivalence of triads

$$\gamma = (\mathrm{proj} \circ i) \times j : (W; \partial_0 W, \partial_1 W) \to (F \times [0,1]; F \times \{0\}, F \times \{1\}).$$

Therefore,

$$\tau(h_1) = \tau(\gamma|\partial_1 W : \partial_1 W \to F \times \{1\}) = \tau\beta(h) - (-1)^n \overline{\tau\beta(h)} \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F),$$

where $\overline{\cdot}$ is induced from the standard involution on $\mathbb{Z}\pi_1 F$. Although the composition

$$\pi_0 \,\mathrm{TOP}^b(F \times \mathbb{R}) \xrightarrow{\beta} \pi_0 \mathrm{I}h\mathrm{cob}(F) \xrightarrow{\tau} \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$$

need not be a group homomorphism (cf. Ling [39]), it is a crossed homomorphism; i.e., $\tau\beta([h \circ k]) = h_{1\sharp}\tau\beta([k]) + \tau\beta([h])$ for $[h], [k] \in \pi_0 \,\mathrm{TOP}^b(F \times \mathbb{R})$ where $h_{1\sharp}$ is the homomorphism induced by the homotopy equivalence $h_1 : F \to F$.

We will need the following version of the Alexander trick in the proof of Proposition 8.4 (cf. Hughes [23, Lemma 6.4]).

Lemma 8.3. *If $h : F \times \mathbb{R} \to F \times \mathbb{R}$ is a bounded homeomorphism such that $h = \mathrm{id}$ on $F \times (-\infty, 0]$, then h is boundedly isotopic to $\mathrm{id}_{F \times \mathbb{R}}$.*

Proof. For $0 \le s < 1$ define $\theta_s : \mathbb{R} \to \mathbb{R}$ by $\theta_s(t) = t - \frac{s}{s-1}$. Define a bounded isotopy $H : h \simeq \mathrm{id}_{F \times \mathbb{R}}$ by

$$H_s = \begin{cases} (\mathrm{id}_F \times \theta_s)^{-1} \circ h \circ (\mathrm{id}_F \times \theta_s), & \text{if } 0 \le s < 1 \\ \mathrm{id}_{F \times \mathbb{R}}, & \text{if } s = 1. \end{cases} \quad \square$$

Proposition 8.4. *If $n = \dim F \ge 4$, then the sequence*

$$\pi_0 \mathrm{TOP}(F) \xrightarrow{\sigma} \pi_0 \,\mathrm{TOP}^b(F \times \mathbb{R}) \xrightarrow{\beta} \pi_0 \mathrm{I}h cob(F) \longrightarrow 0$$

is exact in the sense that β maps the set of cosets of $\pi_0 \,\mathrm{TOP}^b(F \times \mathbb{R}) / \mathrm{Im}(\sigma)$ bijectively onto $\pi_0 \mathrm{I}h cob(F)$; i.e.,

(i) *If $[h_1], [h_2] \in \pi_0 \,\mathrm{TOP}^b(F \times \mathbb{R})$, then $\beta([h_1]) = \beta([h_2])$ if and only if there exists $[g] \in \pi_0 \,\mathrm{TOP}(F)$ such that $[h_2^{-1} h_1] = \sigma([g])$, and*

(ii) *β is surjective.*

Proof. (i) Let $h_i : F \times \mathbb{R} \to F \times \mathbb{R}$ be bounded homeomorphisms for $i = 1, 2$. Choose $L > 0$ such that $h_i(F \times \{L\}) \subseteq F \times (0, \infty)$ so that $W_i = h_i(F \times (-\infty, L]) \backslash F \times (-\infty, 0)$ is an h–cobordism from $F = F \times \{0\}$ to $h_i(F \times \{L\})$ and $\beta([h_i]) = [W_i] \in \pi_0 h\mathrm{cob}(F)$ for $i = 1, 2$. If $[W_1] = [W_2]$, then there exists a homeomorphism $H : W_1 \to W_2$ such that $H| : F \times \{0\} \to F \times \{0\}$ is the identity. In particular, $Hh_1(F \times \{1\}) = h_2(F \times \{L\})$. Let $g : F \to F$ be the homeomorphism defined by $h_2^{-1} H h_1(x, L) = (g(x), 1) \in F \times \{L\}$ for all $x \in F$. Extend H via the identity on $F \times (-\infty, 0]$ to a homeomorphism $\widetilde{H} : (F \times (-\infty, 0]) \cup W_1 \to (F \times (-\infty, 0]) \cup W_2$. Define a hybrid homeomorphism $\widetilde{h} : F \times \mathbb{R} \to F \times \mathbb{R}$ by

$$\widetilde{h}(x, t) = \begin{cases} \widetilde{H} h_1(x, t), & \text{if } t \le L \\ h_2(g(x), t), & \text{if } t \ge L. \end{cases}$$

According to Lemma 8.3 both $\widetilde{h}h_1^{-1}$ and $\widetilde{h}(g^{-1}\times \mathrm{id}_{\mathbb{R}})h_2^{-1}$ are boundedly isotopic to the identity. Thus $\widetilde{h}$ is boundedly isotopic to h_1 and to $h_2(g\times \mathrm{id}_{\mathbb{R}})$ so that $h_2^{-1}h_1$ is boundedly isotopic to $g\times \mathrm{id}_{\mathbb{R}}$ showing $[h_2^{-1}h_1] = \sigma([g])$.

Conversely, if $h_2^{-1}h_1$ is boundedly isotopic to $g\times \mathrm{id}_{\mathbb{R}}$ for some homeomorphism $g : F \to F$, then h_1 is boundedly isotopic to $h_2(g\times \mathrm{id}_{\mathbb{R}})$. If L is large enough, then the isotopy restricts to an isotopy of embeddings carrying $h_1(F\times\{L\})$ onto $h_2(g(F)\times\{L\}) = h_2(F\times\{L\})$ in $F\times(0,\infty)$. The Isotopy Extension Theorem [11] shows that there is an isotopy of $F\times\mathbb{R}$ to itself which is the identity on $F\times(-\infty,0]$ and carries $h_1(F\times\{L\})$ to $h_2(F\times\{L\})$. In particular, there is a homeomorphism $H : W_1 \to W_2$ such that $H|F\times\{0\}$ is the identity. Hence $[W_1] = [W_2] \in \pi_0 h\mathrm{cob}(F)$.
(ii) follows from Ling [39, Prop. 3.2] $\square$

Anderson and Hsiang [2] calculated the homotopy groups of the simplicial set of bounded concordances. We will need their calculation of the group of components.

Proposition 8.5 (Anderson-Hsiang). *If $n = \dim F \geq 5$, then there exists a group isomorphism*

$$\alpha : \pi_0\mathrm{C}^b(F\times\mathbb{R}^i) \longrightarrow \begin{cases} \mathrm{Wh}_1(\mathbb{Z}\pi_1 F), & \text{if } i = 1\\ \widetilde{K}_0(\mathbb{Z}\pi_1 F), & \text{if } i=2\\ K_{2-i}(\mathbb{Z}\pi_1 F), & \text{if } i>2. \quad \square\end{cases}$$

We need to recall the explicit construction of the isomorphism when $i = 1$, $\alpha : \pi_0\mathrm{C}^b(F\times\mathbb{R}) \to \mathrm{Wh}_1(\mathbb{Z}\pi_1F)$. If $h : F\times\mathbb{R}\times[0,1] \to F\times\mathbb{R}\times[0,1]$ is a bounded concordance representing a class $[h] \in \pi_0\mathrm{C}^b(F\times\mathbb{R})$, choose $L > 0$ so large that $h(F\times\{L\}\times[0,1]) \subseteq F\times(0,\infty)\times[0,1]$ and let $W = h(F\times(-\infty,L]\times[0,1])\setminus F\times(-\infty,0)\times[0,1]$, $\partial_0 W = F\times[0,L]\times\{0\}$, and $\partial_1 W = h(F\times(-\infty,L]\times\{1\})\setminus F\times(-\infty,0)\times\{1\}$. Then $(W;\partial_0W,\partial_1W)$ is a relative h–cobordism. In particular, over the boundary of $\partial_0 W$, W restricts to a product h–cobordism

$$(F\times\{0\}\times[0,1]\cup h(F\times\{L\}\times[0,1]; F\times\{0,L\}\times\{0\}, F\times\{0\}\times\{1\}\cup h(F\times\{L\}\times\{1\})).$$

Define $\alpha([h])$ to be the Whitehead torsion $\tau(W,\partial_0W) \in \mathrm{Wh}_1(\mathbb{Z}\pi_1(F\times[0,L]) = \mathrm{Wh}_1(\mathbb{Z}\pi_1F)$.

Recall that $\dim F = n$. Define the *norm homomorphism*

$$N : \mathrm{Wh}_1(\mathbb{Z}\pi_1F) \to \mathrm{Wh}_1(\mathbb{Z}\pi_1F); \qquad x \mapsto x + (-1)^n\overline{x}$$

where $\overline{\cdot}$ is induced from the standard involution on $\mathbb{Z}\pi_1F$.

Proposition 8.6. *If $n = \dim F \geq 5$, then the following diagram commutes*:

$$\begin{array}{ccccc} \pi_0\mathrm{C}^b(F\times\mathbb{R}) & \xrightarrow{\alpha} & \mathrm{Wh}_1(\mathbb{Z}\pi_1F) & \xrightarrow{N} & \mathrm{Wh}_1(\mathbb{Z}\pi_1F)\\ {\scriptstyle\rho}\big\downarrow & & & & \big\uparrow{\scriptstyle\tau}\\ \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}) & \xrightarrow{\beta} & \pi_0\mathrm{I}h\mathit{cob}(F) & \xrightarrow{\subseteq} & \pi_0 h\mathrm{cob}(F). \end{array}$$

Proof. If $[h] \in \pi_0\mathrm{C}^b(F\times\mathbb{R})$ adopt the notation above in the explicit description of α so that $\alpha([h]) = \tau(W,\partial_0W) = x$. For $k = 0,1$ let $i_k : \partial_kW \to W$ denote

the inclusion and $r_k : W \to \partial_k W$ a strong deformation retraction. Then $x = \tau(r_0)$. Since $(W; \partial_0 W, \partial_1 W)$ is a relative h–cobordism between $(n+1)$–dimensional manifolds, it follows that $(r_0 i_1)_* \tau(r_1) = (r_0 i_1)_* \tau(W, \partial_1 W) = (-1)^{n+1}\overline{x}$ by the duality theorem of Milnor [43, p. 394]. Thus, $\tau(i_1) = (-1)^{n+1}\overline{\tau}(i_0) \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 W)$.

Let $j_1 : F \times \{0\} \times \{1\} \to W$ and $j_2 : F \times \{0\} \times \{1\} \to \partial_0 W$ denote the inclusions and let $j_3 : F \times \{0\} \times \{1\} \to \partial_0 W$ be the map $j_3(z, 0, 1) = (z, 0, 0)$. Since $\partial_0 W = F \times [0, L] \times \{0\}$, $\tau(j_3) = 0$.

Since $\tau\beta\rho([h])$ is the Whitehead torsion of $(\partial_1 W, F \times \{0\} \times \{1\})$ in $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$, it suffices to show that

$$i_{1*}\tau(j_2) = \tau(i_0) = (-1)^n \overline{\tau}(i_0) \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 W).$$

The composition formula gives

$$\tau(j_1) = \tau(i_1 j_2) = i_{1*}\tau(j_2) + \tau(i_1).$$

Since $j_1 \simeq i_0 j_3$ and $\tau(j_3) = 0$, the composition formula also gives $\tau(j_1) = \tau(i_0 j_3) = \tau(i_0)$. Thus

$$i_{1*}\tau(j_2) = \tau(i_0) - \tau(i_1) = \tau(i_0) + (-1)^{n+1}\tau(i_0).$$

A similar argument has been used by Siebenmann and Sondow [57, p. 266]. $\square$

Lemma 8.7.
(i) *Suppose there is a diagram*

$$\begin{tikzcd} & A \arrow[d, "\sigma_1"] & \\ A' \arrow[r, "\rho_1"] & B \arrow[r, "\beta"] \arrow[d, "\sigma_2"] & C \\ & C' & \end{tikzcd}$$

such that

(1) A, B, A', C' *are groups (written additively),* C *is a set, and* $\sigma_1, \rho_1, \sigma_2$ *are group homomorphisms,*

(2) $A \xrightarrow{\sigma_1} B \xrightarrow{\beta} C$ *is exact in the sense that* β *is surjective, and if* $b_1, b_2 \in B$ *then* $\beta(b_1) = \beta(b_2)$ *if and only if* $b_2 - b_1 = \sigma_1(a)$ *for some* $a \in A$,

(3) $A' \xrightarrow{\rho_1} B \xrightarrow{\sigma_2} C'$ *is an exact sequence of groups.*

If $b \in B$, *then* $\sigma_2(b) \in \mathrm{Im}(\sigma_2\sigma_1 : A \to C')$ *if and only if* $\beta(b) \in \mathrm{Im}(\beta\rho_1 : A' \to C)$.
(ii) *Suppose further that the diagram above is extended to a diagram*

$$\begin{tikzcd} & A \arrow[d, "\sigma_1"] & & \\ A' \arrow[r, "\rho_1"] & B \arrow[r, "\beta"] \arrow[d, "\sigma_2"] & C \arrow[r, "\tau"] & W \\ D \arrow[r, "\rho_2"] & C' \arrow[d, "\sigma_3"] & & \\ & E & & \end{tikzcd}$$

such that

(1) A, B, A', C', W, D, E *are abelain groups and* ρ_2, σ_3 *are group homomorphisms,*
(2) $\tau : C \to W$ *is a set inclusion,*
(3) *there is a* B*-module structure on* W *which satisifies: if* $b_1, b_2 \in B$ *and* $\sigma_2(b_1) = \sigma_2(b_2)$*, then* $b_1 w = b_2 w$ *for all* $w \in W$*,*
(4) $\tau\beta : B \to W$ *is a crossed homomorphism with respect to the* B*-module structure (i.e.,* $\tau\beta(b_1 + b_2) = b_1\tau\beta(b_2) + \tau\beta(b_1)$ *for all* $b_1, b_2 \in B$*),*
(5) $\tau\beta(-b) = -\tau\beta(b)$ *for all* $b \in B$*,*
(6) $N = \tau\beta\rho_1 : A' \to W$ *is a homomorphism,*
(7) $D \xrightarrow{\rho_2} C' \xrightarrow{\sigma_3} E$ *is an exact sequence of groups.*

There exists a function $\tilde{\beta} : \mathrm{Im}(\sigma_2) \to W/\mathrm{Im}(N)$ *such that if* $b \in B$*, then* $\sigma_3\sigma_2(b) \in \mathrm{Im}(\sigma_3\sigma_2\sigma_1)$ *if and only if the class of* $\tau\beta(b)$ *in* $W/\mathrm{Im}(N)$ *is in* $\tilde{\beta}[\mathrm{Im}(\sigma_2) \cap \mathrm{Im}(\rho_2)]$*.*

Proof. (i) Suppose first that $\sigma_2(b) = \sigma_2\sigma_1(a)$ for some $a \in A$. Then the exact sequence of groups implies that there exists $a' \in A'$ such that $\rho_1(a') = b + \sigma_1(-a)$. Thus $-b + \rho_1(a') = \sigma_1(-a)$ and exactness of the other sequence implies $\beta(b) = \beta\rho_1(a')$.
Conversely, suppose $\beta(b) = \beta\rho_1(a')$ for some $a' \in A'$. Exactness implies that $b = \sigma_1(a) + \rho_1(a')$ for some $a \in A$. Thus $\sigma_2(b) = \sigma_2\sigma_1(a) + \sigma_2\rho_1(a') = \sigma_2\sigma_1(a)$.
(ii) Define $\tilde{\beta} : \mathrm{Im}(\sigma_2) \to W/\mathrm{Im}(N)$ by $\tilde{\beta}(x) = \tau\beta\sigma_2^{-1}(x)$. In order to show that $\tilde{\beta}$ is well-defined, suppose that $\sigma_2(y_1) = \sigma_2(y_2)$ and show that $\tau\beta(y_1) - \tau\beta(y_2) \in \mathrm{Im}(N)$. Since $\ker(\sigma_2) = \mathrm{Im}(\rho_1)$, it follows that $\tau\beta(y_1 - y_2) \in \mathrm{Im}(N)$. Now $\tau\beta(y_1 - y_2) - [\tau\beta(y_1) - \tau\beta(y_2)] = (-y_2)\tau\beta(y_1)\tau\beta(-y_2) - \tau\beta(y_1) + \tau\beta(y_2) = (-y_2)\tau\beta(y_1) + \tau\beta(-y_1) = (-y_1)\tau\beta(y_1) + \tau\beta(-y_1) = \tau\beta(y_1 - y_1) = 0$. Thus, $\tau\beta(y_1 - y_2) = \tau\beta(y_1) - \tau\beta(y_2)$ showing $\tilde{\beta}$ is well-defined.
Suppose that $\sigma_3\sigma_2(b) \in \mathrm{Im}(\sigma_3\sigma_2\sigma_1)$, say $a \in A$ with $\sigma_3\sigma_2 b = \sigma_3\sigma_2\sigma_1(a)$. Then $\sigma_2(b) - \sigma_2\sigma_1(a) \in \ker(\sigma_3 = \mathrm{Im}\,\rho_2$, so let $d \in D$ with $\rho_2(d) = \sigma_2(b) - \sigma_2\sigma_1(a)$. Thus, $\sigma_2(b) = \rho_2(d) + \sigma_2\sigma_1(a)$ and $\rho_2(d) \in \mathrm{Im}(\sigma_2) \cap \mathrm{Im}(\rho_2)$. It follows that $\tilde{\beta}\rho_2(d) = \tau\beta\sigma_2^{-1}(\rho_2(d)) = \tau\beta(b - \sigma_1(a)) = \tau\beta(b) - \tau\beta(\sigma_1(a))$. Thus, we will be done by showing that $\tau\beta(\sigma_1(a) \in \mathrm{Im}(N)$. By part (i), this is equivalent to showing that $\sigma_2\sigma_1(a) \in \mathrm{Im}(\sigma_2\sigma_1)$, which is obviously true.
Conversely, if the class of $\tau\beta(b)$ in $W/\mathrm{Im}(N)$ is in $\tilde{\beta}[\mathrm{Im}(\sigma_2) \cap \mathrm{Im}(\rho_2)]$, choose $x \in \mathrm{Im}(\sigma_2) \cap \mathrm{Im}(\rho_2)$ such that $\tilde{\beta}(x) = \tau\beta(b) + \mathrm{Im}(N)$. Thus, there exists $y \in B$ such that $\sigma_2(y) = x$ and $\tau\beta(b) - \tau\beta(y) \in \mathrm{Im}(N)$. By exactness of $A \xrightarrow{\sigma_1} B \xrightarrow{\beta} C$ there exists $a \in A$ such that $\sigma_1(a) = b - y$, from which it follows that $\sigma_3\sigma_2\sigma_1(a) = \sigma_3\sigma_2(b) - \sigma_3\sigma_2(y) = \sigma_3\sigma_2(b) - \sigma_3\sigma_3(x)$. But $x \in \mathrm{Im}(\rho_2) = \ker(\sigma_3)$, so $\sigma_3\sigma_2\sigma_1(a) = \sigma_3\sigma_2(b)$. □

We now recall the classical classification of fibre bundles over $S^1 \times \mathbb{R}^i$ with fibre F, the classification of manifold approximate fibrations over $S^1 \times \mathbb{R}^i$ with fibre germ $F \times \mathbb{R}^{i+1} \to \mathbb{R}^{i+1}$, and the relationship between these two classifications from Hughes-Taylor-Williams [29], [30]. Let $\mathrm{Bun}(S^1 \times \mathbb{R}^i)_F$ denote the simplicial set of fibre bundles over $S^1 \times \mathbb{R}^i$ with fibre F, so that there exists a homotopy equivalence $\mathrm{Bun}(S^1 \times \mathbb{R}^i)_F \simeq \mathrm{Map}(S^1, \mathrm{BTOP}(F))$. Since $\pi_0 \mathrm{Map}(S^1, \mathrm{BTOP}(F)) = \pi_0 \mathrm{TOP}(F)$, there is a classifying isomorphism $c_1 : \pi_0\mathrm{Bun}(S^1 \times \mathbb{R}^i)_F \to \pi_0 \mathrm{TOP}(F)$. Let $\mathrm{MAF}(S^1 \times \mathbb{R}^i)_{F\times\mathbb{R}^{i+1}}$ denote the simplicial set of manifold approximate fibrations over $S^1 \times \mathbb{R}^i$ with fibre germ the projection $F \times \mathbb{R}^{i+1} \to \mathbb{R}^{i+1}$ and assume $\dim F + i \geq 4$. Since $S^1 \times \mathbb{R}^i$ is parallelizable it follows from [29] that there is a

homotopy equivalence $\mathrm{MAF}(S^1\times\mathbb{R}^i)_{F\times\mathbb{R}^{i+1}}\simeq \mathrm{Map}(S^1,\mathrm{BTOP}^c(F\times\mathbb{R}^{i+1}))$ where $\mathrm{TOP}^c(F\times\mathbb{R}^{i+1})$ denotes the simplicial group of controlled homeomorphisms on $F\times\mathbb{R}^{i+1}$. Since $\mathrm{TOP}^c(F\times\mathbb{R}^{i+1}))\simeq \mathrm{TOP}^b(F\times\mathbb{R}^{i+1})$ by Hughes-Taylor-Williams [31] and $\pi_0\,\mathrm{Map}(S^1,\mathrm{BTOP}^b(F\times\mathbb{R}^{i+1}))=\pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^{i+1})$, there is a classifying isomorphism $c_2:\pi_0\,\mathrm{MAF}(S^1\times\mathbb{R}^i)_{F\times\mathbb{R}^{i+1}}\to\pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^{i+1})$.

Proposition 8.8. *If $\dim F+i\geq 4$, then the following diagram commutes*:

$$\begin{tikzcd}
\pi_0\mathrm{Bun}(S^1\times\mathbb{R}^i)_F \arrow[r, "\varphi"] \arrow[d, "c_1"', "\simeq"] & \pi_0\,\mathrm{MAF}(S^1\times\mathbb{R}^i)_{F\times\mathbb{R}^{i+1}} \arrow[d, "\simeq"', "c_2"] \\
\pi_0\mathrm{TOP}(F) \arrow[r, "\sigma"] & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^{i+1})
\end{tikzcd}$$

where φ is the forgetful map and σ is euclidean stabilization $[h]\mapsto[h\times\mathrm{id}_{\mathbb{R}^{i+1}}]$.

Proof. This follows from Hughes-Taylor-Williams [30, Thm. 0.3]. $\square$

If $p:M\to S^1$ is a manifold approximate fibration with fibre germ $F\times\mathbb{R}\to\mathbb{R}$, then the *monodromy* of p is the class $c_2(p)=[h]\in\pi_0\,\mathrm{TOP}^b(F\times\mathbb{R})$ with $h:F\times\mathbb{R}\to F\times\mathbb{R}$ a bounded homeomorphism. The monodromy induces a well-defined homotopy equivalence $F=F\times\{0\}\xrightarrow{h|}F\times\mathbb{R}\to F$ which in turn induces a homomorphism $h_*:\mathrm{Wh}_1(\mathbb{Z}\pi_1F)\to\mathrm{Wh}_1(\mathbb{Z}\pi_1F)$, also called the monodromy of p.

Theorem 8.9. *Let $p:M\to S^1$ be a manifold approximate fibration with fibre germ $F\times\mathbb{R}\to\mathbb{R}$ and monodromy $[h]$ with $n=\dim F\geq 4$.*

(i) *The following are equivalent*:

(1) *p is controlled homeomorphic to a fibre bundle projection with fibre F.*
(2) $\tau\beta(c_2([p]))=\tau\beta([h])=0\in\mathrm{Wh}_1(\mathbb{Z}\pi_1F)$.

(ii) *The following are equivalent*:

(1) *$p\times\mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection with fibre F.*
(2) $\tau\beta(c_2([p]))=\tau\beta([h])\in\mathrm{Im}\,N\subseteq\mathrm{Wh}_1(\mathbb{Z}\pi_1F)$.

(iii) *There exist a subgroup G of $\widetilde{K}_0(\mathbb{Z}\pi_1F)$ and a function*

$$N_0:G\to\mathrm{Wh}_1(\mathbb{Z}\pi_1F)/\mathrm{Im}\,N$$

such that the following are equivalent:

(1) *$p\times\mathrm{id}_{\mathbb{R}^2}$ is controlled homeomorphic to a fibre bundle projection with fibre F.*
(2) *The class of $\tau\beta(c_2([p]))=\tau\beta([h])$ in $\mathrm{Wh}_1(\mathbb{Z}\pi_1F)/\mathrm{Im}\,N$ is in $N_0(G)$.*

Proof. (i) follows from Propositions 8.4 and 8.8.
(ii) Consider the diagram

$$\begin{tikzcd}
 & \pi_0\,\mathrm{TOP}(F) \arrow[d, "\sigma_1"] & \\
\pi_0\mathrm{C}^b(F\times\mathbb{R}) \arrow[r, "\rho_1"] & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}) \arrow[r, "\beta"] \arrow[d, "\sigma_2"] & \pi_0\mathrm{I}h\mathrm{cob}(F) \\
 & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^2) &
\end{tikzcd}$$

where σ_1, σ_2 denote euclidean stabilization and ρ_1, β have been defined above. According to Proposition 8.8, $p \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle with fibre F if and only if $\sigma_2 c_2([p]) = \sigma_2([h]) \in \mathrm{Im}(\sigma_2\sigma_1)$. By Propositions 8.2, 8.4 and Lemma 8.7, $\sigma_2([h]) \in \mathrm{Im}(\sigma_2\sigma_1)$ if and only if $\beta([h]) \in \mathrm{Im}(\beta\rho_1)$ if and only if $\tau\beta([h]) \in \mathrm{Im}(N)$. Thus, (1) and (2) are equivalent.
(iii) The diagram above can be extended to a diagram

$$\begin{tikzcd}
 & \pi_0\,\mathrm{TOP}(F) \arrow[d, "\sigma_1"] & & \\
\pi_0\mathrm{C}^b(F\times\mathbb{R}) \arrow[r, "\rho_1"] & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}) \arrow[r, "\beta"] \arrow[d, "\sigma_2"] & \pi_0\mathrm{I}h\mathrm{cob}(F) \arrow[r, "\tau"] & \mathrm{Wh}_1(\mathbb{Z}\pi_1F) \\
\pi_0\mathrm{C}^b(F\times\mathbb{R}^2) \arrow[r, "\rho_2"] & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^2) \arrow[d, "\sigma_3"] & & \\
 & \pi_0\,\mathrm{TOP}^b(F\times\mathbb{R}^3) & &
\end{tikzcd}$$

As above, $p \times \mathrm{id}_{\mathbb{R}^2}$ is controlled homeomorphic to a fibre bundle projection with fibre F if and only if $\sigma_3\sigma_2([h]) \in \mathrm{Im}(\sigma_3\sigma_2\sigma_1)$. Since $\pi_0\mathrm{C}^b(F\times\mathbb{R}^2) \cong \widetilde{K}_0(\mathbb{Z}\pi_1F)$ by Proposition 8.5, the result will follow from Lemma 8.7(ii) once it is obsevered that the action of $\pi_0\,\mathrm{TOP}^b(F\times\mathbb{R})$ on $\mathrm{Wh}_1(\mathbb{Z}\pi_1F)$ satisfies items (3) and (5) of 8.7(ii). The first follows from the fact that if $\sigma_2([h]) = \sigma_2([h'])$, then the induced homotopy equivalences $h_1, h_1' : F \to F$ are homotopic and, hence, $h_{1\sharp} = h'_{1\sharp}$. The second follows from the explicit construction of β. $\square$

Remark 8.10. It follows from Hughes-Taylor-Williams [32] that condition 8.9(i)(1) holds if and only if p is homotopic to a fibre bundle projection with fibre F. It seems reasonable to conjecture that condition 8.9(ii)(1) holds if and only if $p \times \mathrm{id}_{\mathbb{R}}$ is properly homotopic to to a fibre bundle projection with fibre F.

We will now prepare for a version of Theorem 8.9(i),(ii) where we allow the fibre of the fibre bundle projections to vary (Theorem 8.13 below). The following result says that we do not have to worry about non-manifold fibres.

Lemma 8.11.

(i) *If $p : M \to S^1$ is a manifold approximate fibration with $m = \dim M \geq 6$ and p is controlled homeomorphic to a bundle projection, then p is controlled homeomorphic to a bundle projection with manifold fibre.*
(ii) *If $p : M \to S^1 \times \mathbb{R}$ is a manifold approximate fibration with $m = \dim M \geq 7$ and p is controlled homeomorphic to a bundle projection, then p is controlled homeomorphic to a bundle projection with manifold fibre.*

Proof. (i) We may assume that $p : M \to S^1$ is a bundle projection. The fibre is a compact ANR X. According to [32] it suffices to show that p is homotopic to a bundle projection with manifold fibre; that is, we need to show that the Farrell fibering obstruction of p vanishes. We will use the version of the total fibering obstruction as exposited in Ranicki [49]. Let $h : X \to X$ be the classical monodromy of p so that the mapping torus $T(h)$ is M. The infinite cyclic cover of M is $X \times \mathbb{R}$ with generating covering translation $\zeta : X \times \mathbb{R} \to X \times \mathbb{R}$; $(x, t) \mapsto (h(x), t+1)$. The

mapping torus $T(\zeta)$ has a preferred finite structure and the fibering obstruction is the torsion of the natural homotopy equivalence $T(\zeta) \to T(h) = M$. The preferred finite structure on $T(\zeta)$ can be defined by choosing a finite CW complex K and a homotopy equivalence $f : K \to X$ (this exists by West [64]). Let $g : X \to K$ be a homotopy inverse for f. Then f, g induce a natural homotopy equivalence $d : K \to X \times \mathbb{R}$; $x \mapsto (f(x), 0)$ and inverse $u : X \times \mathbb{R} \to K$; $(x, t) \mapsto g(x)$. In particular, this is a finite domination of $X \times \mathbb{R}$ so that $T(u\zeta d) \to T(\zeta)$ is the preferred finite structure. Note that $T(u\zeta d) = T(ghf)$ and the composition $T(ghf) \to T(\zeta) \to T(h) = M$ is simple.
(ii) We may assume that $p : M \to S^1 \times \mathbb{R}$ is a bundle projection with fibre a compact ANR X. Let $W = p^{-1}(S^1 \times \{0\})$ and $q = p| : W \to S^1 \times \{0\} = S^1$. Note that $X \times \mathbb{R}^2$ is a manifold since M is a manifold and p is a bundle projection; however, it is unknown whether this implies that $X \times \mathbb{R}$ is a manifold (cf. Daverman [9, Prob. 625]). In particular, W might not be a manifold. On the other hand, $W \times \mathbb{R}$ is homeomorphic to M so that W is resolvable by Quinn [47, 3.2.2]; that is, there exist a manifold N, $\dim N = m - 1 \geq 6$, and a cell–like map $r : N \to W$. It follows as in part (i) that $qr : N \to S^1$ is homotopic to a fibre bundle projection with manifold fibre and hence (by [32]) is controlled homeomorphic to a fibre bundle projection $q' : N \to S^1$ with manifold fibre. Since $p : M \to S^1 \times \mathbb{R}$ is fibre preserving homeomorphic to $q \times \mathrm{id}_{\mathbb{R}} : W \times \mathbb{R} \to S^1 \times \mathbb{R}$, p is controlled homeomorphic to $q \times \mathrm{id}_{\mathbb{R}}$. Siebenmann [56] implies that $r \times \mathrm{id}_{\mathbb{R}} : N \times \mathbb{R} \to W \times \mathbb{R}$ can be arbitrarily closely approximated by homeomorphisms, so that $q \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to $qr \times \mathrm{id}_{\mathbb{R}} : N \times \mathbb{R} \to W \times \mathbb{R}$. Finally, $q \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to $q' \times \mathrm{id}_{\mathbb{R}}$ which is a bundle projection with manifold fibre. □

Lemma 8.12. *Let $p : M \to S^1$ be a manifold approximate fibration with fibre germ $F \times \mathbb{R} \to \mathbb{R}$, monodromy $[h : F \times \mathbb{R} \to F \times \mathbb{R}]$, and $n = \dim F \geq 4$. Suppose F' is a closed manifold for which there is a bounded homeomorphism $k : F \times \mathbb{R} \to F' \times \mathbb{R}$.*

(i) *p is a manifold approximate fibration with fibre germ $F' \times \mathbb{R} \to \mathbb{R}$ and monodromy $[khk^{-1} : F' \times \mathbb{R} \to F' \times \mathbb{R}]$.*
(ii) *If $\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \to \pi_0 \mathrm{I}h cob(F)$ and $\beta' : \pi_0 \mathrm{TOP}^b(F' \times \mathbb{R}) \to \pi_0 \mathrm{I}h cob(F')$ are the 'region between' functions defined above, then there exists $x \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ such that*

$$\tau\beta([h]) = (k^{-1})_* \tau\beta'([khk^{-1}]) + x - h_*(x)$$

where $(k^{-1})_* : \mathrm{Wh}_1(\mathbb{Z}\pi_1 F') \to \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ is induced by the composition $F' = F' \times \{0\} \xrightarrow{k^{-1}|} F \times \mathbb{R} \to F$. Moreover, x is represented by the torsion of the h–cobordism associated to the bounded homeomorphism $k^{-1} : F' \times \mathbb{R} \to F \times \mathbb{R}$.

Proof. (i) If p is considered to have fibre germ $F \times \mathbb{R} \to \mathbb{R}$, then the affect of the classifying map c_2 is to turn $p : M \to S^1$ into a fibre bundle over S^1 with fibre $F \times \mathbb{R}$ and structure group $\mathrm{TOP}^b(F \times \mathbb{R})$. The monodromy h is then the classical monodromy of this bundle. The bundle can be considered to be a bundle with fibre $F' \times \mathbb{R}$, structure group $\mathrm{TOP}^b(F' \times \mathbb{R})$ and monodromy khk^{-1}. See [29], [30].
(ii) Choose $L > 0$ large. Let

$$W = (khk^{-1})(F' \times (-\infty, L]) \setminus F' \times (-\infty, 0) \subseteq F' \times \mathbb{R}$$

so that

$$(W; F' \times \{0\}, khk^{-1}(F' \times \{L\})$$

is an h–cobordism whose torsion is $\tau\beta'([khk^{-1}])$. Let

$$W_k = k(F \times [-L, \infty)) \setminus F' \times (0, \infty) \subseteq F' \times \mathbb{R}$$

so that $(W_k; k(F \times \{-L\}), F' \times \{0\})$ is an h–cobordism. Let

$$W_{k^{-1}} = F \times (-\infty, 2L] \setminus k^{-1}(F' \times (-\infty, L))$$

so that $(W_{k^{-1}}; k^{-1}(F' \times \{L\}), F \times \{2L\})$ is an h–cobordism. Let

$$U = k^{-1}W_k \cup k^{-1}W \cup hW_{k^{-1}} \subseteq F \times \mathbb{R}.$$

Note that $k^{-1}W_k \cap k^{-1}W = k^{-1}(F' \times \{0\})$, $k^{-1}W \cap hW_{k^{-1}} = hk^{-1}(F' \times \{L\})$, $k^{-1}W_k \cap hW_{k^{-1}} = \emptyset$, and that $U = h(F \times (-\infty, 2L]) \setminus F \times (-\infty, -L) \subseteq F \times \mathbb{R}$ so that $(U; F \times \{-L\}, h(F \times \{2L\}))$ is an h–cobordism with torsion $\tau(U, F \times \{-L\}) = \tau\beta([h]) \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$. The standard sum and composition formulae imply that

$$\begin{aligned}&\tau(U, F \times \{-L\})\\ &= \tau(k^{-1}W_k, F \times \{-L\}) + (k^{-1})_*\tau(W, F' \times \{0\}) + h_*(k^{-1})_*\tau(kW_{k^{-1}}, F' \times \{L\}).\end{aligned}$$

Let $x = \tau(k^{-1}W_k, F \times \{-L\}) \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$. It is easy to see that

$$x + (k^{-1})_*\tau(kW_{k^{-1}}, F' \times \{L\}) = 0$$

so that $\tau(U, F \times \{-L\}) = x + (k^{-1})_*\tau(W, F' \times \{0\}) - h_*(x)$. □

Theorem 8.13. *Let $p : M \to S^1$ be a manifold approximate fibration with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ and monodromy $[h]$.*
(i) *If $n = \dim F \geq 5$, then the following are equivalent*:
  (1) *p is controlled homeomorphic to a fibre bundle projection.*
  (2) $\tau\beta(c_2([p])) = \tau\beta([h]) = 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/\operatorname{Im}(1 - h_*)$.
(ii) *If $n = \dim F \geq 6$, then the following are equivalent*:
  (1) *$p \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection.*
  (2) $\tau\beta(c_2([p])) = \tau\beta([h]) = 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/(\operatorname{Im} N + \operatorname{Im}(1 - h_*))$.

Proof. (i) (1) *implies* (2): By Lemma 8.11(i) we may assume that p is controlled homeomorphic to a bundle projection with fibre a closed manifold F'. By uniqueness of fibre germs [29] there exists a bounded homeomorphism $k : F \times \mathbb{R} \to F' \times \mathbb{R}$. An application of Theorem 8.9(i) with F' replacing F implies that $\tau\beta'(c_2[p]) = 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F')$. Now Lemma 8.12(ii) implies that $\tau\beta(c_2[p]) = \tau\beta([h]) = x - h_*(x)$ for some $x \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$.
(2) *implies* (1): If $\tau\beta([h]) = x - h_*(x)$ for some $x \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$, choose an h–cobordism $(W; F, F')$ such that $x = \tau(W, F)$. In fact, there is a bounded homeomorphism $k : F \times \mathbb{R} \to F' \times \mathbb{R}$ such that $W = k^{-1}(F' \times (-\infty, L]) \setminus F \times (-\infty, 0)$ for some large $L > 0$ (this is the h–cobordism associated to k^{-1}). Lemma 8.12(i) implies that p is a manifold approximate fibration with fibre germ $F' \times \mathbb{R} \to \mathbb{R}$ and monodromy khk^{-1}. It follows from Lemma 8.12(ii) that $(k^{-1})_*\tau\beta'([khk^{-1}]) = 0$.

Hence $\tau\beta'([khk^{-1}]) = 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F')$. Finally, Theorem 8.9(i) implies that p is controlled homeomorphic to a bundle projection with fibre F'.
(ii) (1) *implies* (2): By Lemma 8.11(ii) we may assume that $p \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a bundle projection with fibre a closed manifold F'. As in (i) there exists a bounded homeomorphism $k : F \times \mathbb{R} \to F' \times \mathbb{R}$. By Lemma 8.12(ii) there exists $x \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ such that $\tau\beta([h]) = k_*^{-1}\tau\beta'([khk^{-1}]) + x - h_*(x)$. Lemma 8.12(i) implies that p is a manifold approximate fibration with fibre germ $F' \times \mathbb{R} \to \mathbb{R}$ and monodromy $[khk^{-1}]$. Since $p \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection with fibre F', Theorem 8.9 implies that $\tau\beta'([khk^{-1}]) = N'z$ for some $z \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F')$ where $N' : \mathrm{Wh}_1(\mathbb{Z}\pi_1 F') \to \mathrm{Wh}_1(\mathbb{Z}\pi_1 F')$ is the norm map. Thus $\tau\beta([h]) = k_*^{-1}N'z + x - h_*(x) = Nk_*^{-1}z + x - h_*x$.
(2) *implies* (1): Suppose $\tau\beta([h]) = Nz + x - h_*x$. As in (i) there exist a closed manifold F' and a bounded homeomorphism $k : F \times \mathbb{R} \to F' \times \mathbb{R}$ such that x is represented by the torsion associated to k^{-1} via the 'region between' construction. Lemma 8.12(ii) implies that $\tau\beta([h]) = k_*^{-1}\tau\beta'([khk^{-1}]) + x - h_*x$. Hence, $k_*^{-1}\tau\beta'([khk^{-1}]) = Nz$ and $\tau\beta'([khk^{-1}]) = k_*Nz = N'k_*^{-1}z$. Since p is a manifold approximate fibration with fibre germ $F' \times \mathbb{R} \to \mathbb{R}$ and monodromy khk^{-1}, Theorem 8.9 implies that $p \times \mathrm{id}_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle with fibre F'. □

Remark 8.14.

(i) As in Remark 8.10 it follows from Hughes-Taylor-Williams [32] that condition 8.13(i)(1) holds if and only if p is homotopic to a fibre bundle projection. It seems reasonable to conjecture that condition 8.13(ii)(1) holds if and only if $p \times \mathrm{id}_{\mathbb{R}}$ is properly homotopic to to a fibre bundle projection.
(ii) Another way to prove 8.13(i) is to identify $\tau\beta(c_2([p]))$ with the Farrell fibering obstruction of p.

Let $\mathbb{Z}_q$ denote the finite cyclic group of order q.

Proposition 8.15.

(i) *If* $\pi_1(F) = \mathbb{Z}_q$, $q > 3$ *is prime, and* $\dim F = n \geq 6$ *is even, then* $N : \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q]) \to \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ *is not surjective, but* $\tau\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \to \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ *is surjective.*
(ii) *If* $n \geq 5$ *is odd and* $q > 3$ *is prime, then there exists a closed manifold* F *such that* $\dim F = n$, $\pi_1(F) = \mathbb{Z}_q$ *and* $N : \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q]) \to \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ *is not surjective (in fact, it is the* 0 *homomorphism), but* $\tau\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \to \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ *is surjective.*
(iii) *If* q *is prime, then* $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_q])$ *is a finite group and* $K_{-i}(\mathbb{Z}[\mathbb{Z}_q]) = 0$ *for all* $i > 0$. *If, in addition,* $5 \leq q \leq 19$, *then* $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_q]) = 0$.

Proof. Let $q > 3$ be a prime number. It is known that $\mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ is free abelian of finite non-zero rank and the standard involution $\bar{\cdot}$ acts by the identity (Bass [4], Bass-Milnor-Serre [5], Wall [60]; see Oliver [46] for an exposition). Let F be a closed manifold with $\dim F = n \geq 5$ and $\pi_1(F) = \mathbb{Z}_q$. Then $N : \mathrm{Wh}_1(\mathbb{Z}\pi_1 F) \to \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ is multiplication by 2 if n is even and multiplication by 0 if n is odd, and therefore not surjective. According to Lawson [38], if n is even, $\pi_0 \mathrm{I}h\mathrm{cob}(F) = \pi_0 h\mathrm{cob}(F)$, and if n is odd, there exist manifolds F as above such that $\pi_0 \mathrm{I}h\mathrm{cob}(F) = \pi_0 h\mathrm{cob}(F)$. Since Proposition 8.4 implies that $\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \to \pi_0 \mathrm{I}h\mathrm{cob}(F) =$

$\pi_0 h\mathrm{cob}(F)$ is surjective, it follows that $\tau\beta : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}) \to \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_q])$ is surjective. This proves (i) and (ii).
For (iii) see Rosenberg [50, pp. 23, 157]. □

Theorem 8.16.

(i) *If q is a prime, $5 \leq q \leq 19$, $n \geq 6$ is even, F is any closed manifold with $\pi_1(F) = \mathbb{Z}_q$ and $\dim F = n$, then there exists a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ such that for all $i \geq 0$, $p \times \mathrm{id}_{\mathbb{R}^i} : M \times \mathbb{R}^i \to S^1 \times \mathbb{R}^i$ is not controlled homeomorphic to a fibre bundle projection with fibre F.*

(ii) *If $q > 3$ is prime, $n \geq 5$ is odd, then there exists a closed manifold F with $\pi_1(F) = \mathbb{Z}_q$ and $\dim F = n$ and a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ such that for all $i \geq 0$, $p \times \mathrm{id}_{\mathbb{R}^i} : M \times \mathbb{R}^i \to S^1 \times \mathbb{R}^i$ is not controlled homeomorphic to a fibre bundle projection with fibre F.*

(iii) *If $n \geq 6$ is even, F is any closed manifold with $\pi_1(F) = \mathbb{Z}_5$ and $\dim F = n$, then there exists a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ such that for all $i \geq 0$, $p \times \mathrm{id}_{\mathbb{R}^i} : M \times \mathbb{R}^i \to S^1 \times \mathbb{R}^i$ is not controlled homeomorphic to a fibre bundle projection.*

(iv) *If $n \geq 5$ is odd, then there exists a closed manifold F with $\pi_1(F) = \mathbb{Z}_5$ and $\dim F = n$ and a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ such that for all $i \geq 0$, $p \times \mathrm{id}_{\mathbb{R}^i} : M \times \mathbb{R}^i \to S^1 \times \mathbb{R}^i$ is not controlled homeomorphic to a fibre bundle projection.*

Proof. (i) According to Proposition 8.8 we need a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ and monodromy $[h] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R})$ such that for all $i \geq 0$, $[h \times \mathrm{id}_{\mathbb{R}^i}] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{i+1})$ is not in $\mathrm{Im}(\sigma : \pi_0 \mathrm{TOP}(F) \to \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{i+1}))$. According to Propositions 8.2, 8.5, and 8.15(iii), $\sigma : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{2+i}) \to \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{3+i})$ is injective for all $i \geq 0$. Hence, it suffices to find a manifold approximate fibration p with monodromy $[h] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R})$ such that $[h \times \mathrm{id}_{\mathbb{R}}] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^2)$ is not in $\mathrm{Im}(\sigma : \pi_0 \mathrm{TOP}(F) \to \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^2))$; that is, such that $p \times \mathrm{id}_{\mathbb{R}}$ is not controlled homeomorphic to a fibre bundle projection with fibre F. According to Theorem 8.9(ii) this is equivalent to $\tau\beta([h]) \neq 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/\mathrm{Im}N$. Such monodromies exist by Proposition 8.15(i).

(ii) This is similar to (i) except now we know only that $\sigma : \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{3+i}) \to \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^{4+i})$ is injective for all $i \geq 0$. Hence, it suffices to find a manifold approximate fibration p with monodromy $[h] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R})$ such that $[h \times \mathrm{id}_{\mathbb{R}^2}] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^3)$ is not in $\mathrm{Im}(\sigma : \pi_0 \mathrm{TOP}(F) \to \pi_0 \mathrm{TOP}^b(F \times \mathbb{R}^3))$; that is, such that $p \times \mathrm{id}_{\mathbb{R}^2}$ is not controlled homeomorphic to a fibre bundle projection with fibre F. According to Proposition 8.15(ii),(iii) $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/\mathrm{Im}N = \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)$ and is infinite (cf. proof of 8.15). Hence, since 8.15(iii) implies that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_q])$ is finite, the result follows from Proposition 8.9(iii).

(iii) As in (i) it suffices to find a manifold approximate fibration $p : M \to S^1$ with fibre germ $F \times \mathbb{R} \to \mathbb{R}$ and monodromy $[h] \in \pi_0 \mathrm{TOP}^b(F \times \mathbb{R})$ such that $p \times \mathrm{id}_{\mathbb{R}}$ is not controlled homeomorphic to a fibre bundle projection. According to Theorem 8.13(ii) this is equivalent to $\tau\beta([h]) \neq 0 \in \mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/(\mathrm{Im}N + \mathrm{Im}(1 - h_*))$. But $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F) = \mathrm{Wh}_1(\mathbb{Z}[\mathbb{Z}_5])$ is isomorphic to $\mathbb{Z}$ so that $h_* = \pm 1$ and $1 - h_* = 0, 2$.

As noted in the proof of Proposition 8.15, $N = 0$ so that $\mathrm{Wh}_1(\mathbb{Z}\pi_1 F)/(\mathrm{Im}N + \mathrm{Im}(1-h_*)) \neq 0$ and the result follows from Proposition 8.13(ii).
(iv) is similar to (iii). □

Proof of Theorem 8.1. Let X be the open mapping cylinder of a manifold approximate fibration $p : M \to S^1$ constructed in Theorem 8.16(iii) or (iv). If $S^1 \times \mathbb{R}^i$ had a fibre bundle mapping cylinder neighborhood in $X \times \mathbb{R}^i$, then according to Theorem 2.2, $p \times \mathrm{id}_{\mathbb{R}^{i+1}}$ would be controlled homeomorphic to a fibre bundle projection, contradicting Theorem 8.16. Since block bundles with fibre F are classified by $\mathrm{B}\widetilde{\mathrm{TOP}}(F)$, equivalence classes of block bundles over $S^1 \times \mathbb{R}^i$ correspond to $\pi_0\widetilde{\mathrm{TOP}}(F)$. Since $\pi_0\mathrm{TOP}(F) \to \pi_0\widetilde{\mathrm{TOP}}(F)$ is surjective, the result on block bundles follows from the fibre bundle case. □

Remark 8.17. (i) If (X, B) is a manifold stratified pair, it is the case that for a large enough torus, the quotient $X \times T/(B \times T = B)$ does have a block structure. Moreover, the block structure on the 'links' is not arbitrary: it has some nice transfer invariance properties. In other words, for each simplex Δ of B one has a nice manifold which maps to $\Delta \times T$, with control in the T direction. (What we have shown here is that one cannot block over simplices of $B \times T$.) This structure is called a STIBB[1] in [62] and is applied there to give a stable surgery exact sequence for stratified spaces. Indeed, if one had block structures stably then the L–cosheaves in the stable classification theorem [62, §6.2] would have to have the 's' decoration (as in the 'PT category' in [62, §6.1]) rather than the $-\infty$ decoration that arises. The differences between these decorations are accounted for by Tate cohomology calculations rather similar to those done here.

It is not too difficult to combine Theorem 2.2 with the classification theorem of [29], and the stabilization theorem of [63] to give a proof of the stable classification theorem for $S^{-\infty}(X \mathrm{rel}\ B)$. Using [29] the stable germ neighborhoods are computed by maps $[B, \mathrm{BTOP}^b(F \times E)]$ which is the same as $[B, \mathrm{B}\widetilde{\mathrm{Top}}^b(F \times E)]$ by [63], the last of which is computed by bounded block surgery using $L^{-\infty}(\text{holink})$. Different structures with the same germ near the singular stratum can the be compared using ordinary rel ∞ surgery on the complement. The result of this analysis is just a Poincaré duality away from the result as expressed in [62].
(ii) These examples are closely related to those constructed by Anderson [1].
(iii) Husch [33] used nontrivial inertial h–cobordisms to construct exotic manifold approximate fibrations over S^1.
(iv) Using the the tables for relative class numbers in Washington [61, p. 412], it is possible to construct a few more even dimensional manifolds as in Theorem 8.16(i) for primes q with $3 < q < 67$. We don't know of other calculations which give more manifolds as in Theorem 8.16(iii) and (iv).

9. Extensions of isotopies and h–cobordisms

In this section we combine the geometry of teardrop neighborhoods with manifold approximate fibration theory in order to prove parametrized isotopy extension and h–cobordism extension theorems for manifold stratified pairs.

[1] An equivalent notion is used by Yan in [68]: one has blocks over $\Delta \times E$ where E is an Euclidean space, and the data is bounded in the E direction.

Extending isotopies.

Proof of Corollary 2.4 (*Parametrized Isotopy Extension*). Let (X,B) be a manifold stratified pair with $\dim X \geq 5$ and B a closed manifold. Suppose $h : B\times\Delta^k \to B\times\Delta^k$ is a k–parameter isotopy (in particular, $h|B\times\{0\} = \mathrm{id}_{B\times\{0\}}$). We are required to find a k–parameter isotopy $\tilde{h} : X\times\Delta^k \to X\times\Delta^k$ extending h which is supported in a given neighborhood of B. Since B has a teardrop neighborhood in X (Theorem 2.1) there exist an open neighborhood U of B in X (which we can take to be contained in the given neighborhood of B) and a proper map $f : U \to B\times(-\infty,+\infty]$ such that $f| : B \to B\times\{+\infty\}$ is the identity and $f| : U\setminus B \to B\times B\times\mathbb{R}$ is a manifold approximate fibration. We consider Δ^k embedded as a convex subspace of $\mathbb{R}^k$ with the origin the zeroth vertex (basepoint) of Δ^k. Define a k–parameter isotopy $g : B\times\mathbb{R}\times\Delta^k \to B\times\mathbb{R}\times\Delta^k$ by letting $g_t : B\times\mathbb{R} \to B\times\mathbb{R}, t\in\Delta^k$, be given by

$$g_t(x,s) = \begin{cases} (h_t(x),s), & \text{if } s\geq 0 \\ (h_{(1+s)t}(x),s), & \text{if } -1\leq s\leq 0 \\ (x,s), & \text{if } s\leq -1. \end{cases}$$

Let $\mathcal{U}$ be an open cover of $B\times\mathbb{R}$ whose mesh goes to 0 near $B\times\{+\infty\}$; i.e., if $V\in\mathcal{U}$ and $V\cap(B\times[N,+\infty)\neq\emptyset$ then $\operatorname{diam} V < \frac{1}{N}$ for $N = 1,2,3,\dots$ (cf. the definition of Ψ in §5). By the Approximate Isotopy Covering Theorem for manifold approximate fibrations (see [28, 17.4] for information on how this follows from [24]) there exists a k–parameter isotopy $\tilde{g} : (U\setminus B)\times\Delta^k \to (U\setminus B)\times\Delta^k$ such that for each $t\in\Delta^k$

(1) $f\tilde{g}_t$ is $\mathcal{U}$–close to $g_tf|(U\setminus B)$, and
(2) $\tilde{g}_t|f^{-1}(B\times(-\infty,-2]) =$ the inclusion.

Finally, define $\tilde{h}_t : X\to X$, $t\in\Delta^k$, by

$$\tilde{h}_t = \begin{cases} h_t, & \text{on } B \\ \tilde{g}_t, & \text{on } U\setminus B \\ \mathrm{id}_{X\setminus U} & \text{on } X\setminus U. \quad\square \end{cases}$$

Stratified h-cobordisms. Throughout the rest of this section we let (X,B) be a fixed manifold stratified pair with B a closed manifold with $\dim B\geq 5$. We now define stratified h–cobordisms. The definition is a bit more complicated than in [48] because we have not allowed manifold strata to have boundaries.

Definition 9.1. A *stratified h–cobordism on* (X,B) is denoted $(\widetilde{W};\partial_0\widetilde{W},\partial_1\widetilde{W})$ and consists of a homotopically stratified pair $(\widetilde{W},W)$ with finitely dominated local holinks such that
(i) $\widetilde{W}$ is a locally compact separable metric space,
(ii) there is an h–cobordism $(W;\partial_0W,\partial_1W)$ with $\partial_0W = B$,
(iii) there are disjoint closed subspaces $\partial_0\widetilde{W},\partial_1\widetilde{W}\subseteq\widetilde{W}$ with $X = \partial_0\widetilde{W}$ satisfying:

- (a) $\partial_i\widetilde{W}\cap W = \partial_iW$ for $i = 0,1$,
- (b) $\widetilde{W}\setminus W$ is a manifold with boundary $(\partial_0\widetilde{W}\setminus\partial_0W)\cup(\partial_1\widetilde{W}\setminus\partial_1W)$,
- (c) $\partial_i\widetilde{W}$ is a stratum preserving proper strong deformation retract of $\widetilde{W}$ for $i = 0,1$.

The stratified h–cobordism $(\widetilde{W};\partial_0\widetilde{W},\partial_1\widetilde{W})$ is said to *extend* the h–cobordism $(W;\partial_0 W,\partial_1 W)$. Note that $(\widetilde{W}\backslash W;\partial_0\widetilde{W}\backslash\partial_0 W,\partial_1\widetilde{W}\backslash\partial_1 W)$ is a proper h–cobordism on $\partial_0\widetilde{W}\setminus\partial_0 W$.

The following result is not needed in the rest of this section, but is included to show that stratified h-cobordisms keep one inside the category of manifold stratified pairs.

Proposition 9.2. *If $(\widetilde{W};\partial_0\widetilde{W},\partial_1\widetilde{W})$ is a stratified h-cobordism on (X,B) extending the h-cobordism $(W;\partial_0 W,\partial_1 W)$ on B, then $(\partial_1\widetilde{W},\partial_1 W)$ is a manifold stratified pair.*

Proof. By definition $(\widetilde{W},W)$ is a homotopically stratified pair with finitely dominated local holinks. Of course, $\partial_1 W$ and $\partial_1\widetilde{W}\setminus\partial_1 W$ are manifolds. The forward tameness of $\partial_1 W$ in $\partial_1\widetilde{W}$ follows from the facts that W is forward tame in $\widetilde{W}$ and $\partial_1\widetilde{W}$ is a stratum preserving retract of $\widetilde{W}$. Moreover, since $q:\mathrm{holink}(\widetilde{W},W)\to W$ is a fibration with finitely dominated fibre and a stratum preserving strong deformation of $\widetilde{W}$ to $\partial_1\widetilde{W}$ induces a strong deformation retraction of $\mathrm{holink}(\widetilde{W},W)$ to $\mathrm{holink}(\partial\widetilde{W},\partial_1 W)$ which, when restricted to $q^{-1}(\partial_1 W)$ is fibre preserving over $\partial_1 W$, it follows that $\mathrm{holink}(\partial_1\widetilde{W},\partial_1 W)\to\partial_1 W$ is a fibration with finitely dominated fibre. □

We now fix some notation which will be used throughout the rest of this section.

Notation 9.3. Since B has a teardrop neighborhood in X (Theorem 2.1) there exist an open neighborhood U of B in X and a proper map $f:U\to B\times(-\infty,+\infty]$ such that $f|:B\to B\times\{+\infty\}$ is the identity and $f|:U\setminus B\to B\times\mathbb{R}$ is a manifold approximate fibration.

Definition 9.4. An *h-cobordism on X rel B* consists of:
(i) a proper h-cobordism $(V;\partial_0 V,\partial_1 V)$ on $\partial_0 V = X\setminus B$ (in particular, $\partial_i V$ is a proper strong deformation retract of V for $i=0,1$),
(ii) a map of triads

$$g:(N;\partial_0 N,\partial_1 N)\to(B\times\mathbb{R}\times[0,1];B\times\mathbb{R}\times\{0\},B\times\mathbb{R}\times\{1\})$$

where:

(a) N is an open subset of V and is a neighborhood of the end of V determined by B (i.e., for a proper retraction $r:V\to X\setminus B$ there exists a neighborhood U' of B in X such that $r^{-1}(U'\setminus B)\subseteq N$),
(b) $\partial_i N = N\cap\partial_i V$ for $i=0,1$,
(c) g is a proper approximate fibration,
(d) $\partial_0 N = U$,
(e) $g|\partial_0 N = f$.

Here is some explanation for this definition.

Remarks 9.5.

(1) The teardrop $V\cup_g(B\times[0,1])$ contains $X=\partial_0 V\cup_{g|}B\times\{0\}$ so that the triad

$$(V\cup_g B\times[0,1];X,\partial_1\cup_{g|}B)$$

is a stratified h-cobordism on (X, B) extending the trivial h-cobordism on B. The fact that the properties of Definition 9.1 are indeed satisfied is a special case of Theorem 9.6 below. This is why $(V; \partial_0 V, \partial_1 V)$ is called an h-cobordism on X rel B: because V can be compactifed (if X is compact) by adding $B \times [0,1]$ to obtain a stratified h-cobordism on (X, B) which is trivial on B.

(2) Suppose $(\widetilde{W}; \partial_0 \widetilde{W}, \partial_1 \widetilde{W})$ is any stratified h-cobordism on (X, B) extending $(W; \partial_0 W, \partial_1 W)$. It follows that $(\widetilde{W} \setminus W; \partial_0 \widetilde{W} \setminus \partial_0 W, \partial_1 \widetilde{W} \setminus \partial_1 W)$ is an h-cobordism on X rel B. As noted above, this is obviously a proper h-cobordims on $X \setminus B$. A proof of the other properties in Definition 9.4 requires the advanced teardrop technology from [26],[27] (because $\widetilde{W}$ has more than two strata). Likewise, using this advanced teardrop technology we will be able to reformulate Definition 9.4 to be more along the lines of Definition 9.1. It is because [27] has not yet appeared that we are taking the current approach.

(3) A simple example of an h-cobordism on X rel B is the trivial one $((X \setminus B) \times [0,1]; X \setminus B \times \{0\}, X \setminus B \times \{1\})$. For the open set $N \subseteq (X \setminus B) \times [0,1]$ in Definition 9.4(ii) we take $(U \setminus B) \times [0,1]$. Thus, the Teardrop Neighborhood Existence Theorem 2.1 is required to show that the trivial h-cobordism is an example. Theorem 9.6 below, when applied to this trivial h-cobordism, is nevertheless non-trivial. This special case (stated as Corollary 9.7) best illustrates the power of the techniques of the current paper without making motivational appeal to advanced teardrop technology.

The next result shows how teardrop technology can be used to extend an h-cobordism on B to a teardrop neighborhood of B in X. Moreover, the extension can be chosen so that on the complement of B, it is any given h-cobordism on X rel B. The key fact that makes teardrop technology applicable to this problem is that h-cobordisms on B become trivial h-cobordisms on $B \times \mathbb{R}$ after crossing with $\mathbb{R}$.

Theorem 9.6. *Let (X, B) be a manifold stratified pair with B a closed manifold,* $\dim B \geq 5$. *If $(V; \partial_0 V, \partial_1 V)$ is an h-cobordism on X rel B and $(W; \partial_0 W, \partial_1 W)$ is an h-cobordism on B, then there exists a stratified h-cobordism $(\widetilde{W}; \partial_0 \widetilde{W}, \partial_1 \widetilde{W})$ extending $(W; \partial_0 W, \partial_1 W)$ such that*

$$(\widetilde{W} \setminus W; \partial_0 \widetilde{W} \setminus \partial_0 W, \partial_1 \widetilde{W} \setminus \partial_1 W) = (V; \partial_0 V, \partial_1 V).$$

Proof. As is well-known $(W; \partial_0 W, \partial_1 W) \times \mathbb{R}$ is a trivial h–cobordism; i.e., there exists a homeomorphism $h : W \times \mathbb{R} \to B \times \mathbb{R} \times [0,1]$ such that $h| : \partial_0 W \times \mathbb{R} = B \times \mathbb{R} \to B \times \mathbb{R} \times \{0\}$ is the identity. Let $N \subseteq V$ and $g : N \to B \times \mathbb{R} \times [0,1]$ be as in Definition 9.4. Define $\tilde{f} : N \to W \times \mathbb{R}$ to be the composition

$$\tilde{f} : N \xrightarrow{g} B \times \mathbb{R} \times [0,1] \xrightarrow{h^{-1}} W \times \mathbb{R}.$$

Form the teardrop $\widetilde{W} = V \cup_{\tilde{f}} W$. The pair $(\widetilde{W}, W)$ is homotopically stratified with finitely dominated local holinks and $\widetilde{W}$ is a locally compact separable metric space by Corollary 4.10. Let $\partial_i \widetilde{W} = \partial_i V \cup_{g|} B \times \{i\}$ for $i = 0, 1$ which clearly are disjoint closed subsets of $\widetilde{W}$, and $\partial_0 \widetilde{W} = X$. Note that $\widetilde{W} \setminus W = V$ is a manifold with

boundary $\partial_0 V \cup \partial_1 V$ as required. In order to show that $\partial_i \widetilde{W}$ is a stratum preserving strong deformation retract of $\widetilde{W}$ for $i = 0, 1$, one can use the fact that $\partial_i V$ is a strong deformation retract of V together with the homotopy extension theorem, to show that it suffices to define stratum preserving strong deformation retractions on $N \cup_{\tilde{f}} W$. We concentrate on the $i = 0$ case since the $i = 1$ case is similar. Since $\partial_0 W \hookrightarrow W$ is a homotopy equivalence, there exists a strong deformation retraction $r : W \times I \to W$ of W to $\partial_0 W$ (thus, $r_0 = \mathrm{id}_W$, $r_1(W) \subseteq \partial_0 W$ and $r_t|\partial_0 W$ equals the inclusion for $t \in I$). Since $\tilde{f} : N \to W \times \mathbb{R}$ is an approximate fibration, there exists a homotopy $\tilde{r} : N \times I \to N$ such that

(1) $\tilde{r}_0 = \mathrm{id}_N$,
(2) $\tilde{r}_t|\partial_0 N =$ inclusion for each $t \in I$,
(3) $\tilde{r}_1(N) \subseteq \partial_0 N$,
(4) if $(x, s) \in \tilde{f}^{-1}(W \times [k, +\infty)) \subseteq N$ and $k = 1, 2, 3, \ldots$, then for each $t \in I$

$$d(\tilde{f}\tilde{r}(x, s, t), r(\tilde{f}(x, s), t)) < 1/k.$$

(This comes from approximately lifting the homotopy r with very good control near $W \times \{+\infty\}$. To get condition (3), first get a homotopy as above that pulls N close to $\partial_0 N$, in fact, so close that an additional push along a collar will not destroy the estimates in condition (4).) Define $R : N \cup_{\tilde{f}} W \times I \to \widetilde{W}$ by requiring $R|W \times I = r$ and $R|N \times I = \tilde{r}$. The continuity of R follows from Lemma 3.4. $\square$

Corollary 9.7 (h–cobordism Extension). *If $(W; \partial_0 W, \partial_1 W)$ is an h–cobordism with $\partial_0 W = B$, then there exists a stratified h–cobordism $(\widetilde{W}; \partial_0 \widetilde{W}, \partial_1 \widetilde{W})$ with $\partial_0 \widetilde{W} = B$ extending W.*

Proof. This follows immediately from Theorem 9.6. $\square$

Remark 9.8. (i) Quinn [48, 1.8] gives an h–cobordism theorem for stratified spaces. He shows that if a suitable torsion vanishes the h–cobordism is a product, but does not prove there is a realization theorem for torsions (cf. [48, p. 498]). The realization for $\mathrm{Wh}^{\mathrm{top}}(X \mathrm{rel}\ B)$ (the set of equivalence classes of h-cobordisms on X rel B) is a natural extension of the realization of elements of Siebenmann's proper Whitehead group $\mathrm{Wh}^p(W)$ for a noncompact manifold W with a tame end [54]. Indeed the latter is the special case of the former obtained by one point compactifying W (see the picture on p. 132 of [62]). What is missing from [48] then is the proof that $\mathrm{Wh}^{\mathrm{top}}(X) \to \mathrm{Wh}^{\mathrm{top}}(X \mathrm{rel}\ B) \times \mathrm{Wh}(B)$ is surjective (where $\mathrm{Wh}^{\mathrm{top}}(X)$ is the set of equivalence classes of stratified h-cobordisms on X). Theorem 9.6 completes the missing step. F. Connolly and B. Vajiac have recently obtained related results.
(ii) We suspect that there is a fibration of h-cobordism spaces whose fibration sequence at π_0 contains this discusion. We hope to return to this, as well as a discussion of stratified h-cobordisms on manifold stratified spaces with more than two strata, in a later paper.
(iii) Jones [35] proved a concordance extension theorem for locally flat submanifolds of topological manifolds of dimension greater than four. His proof uses manifold approximate fibration techniques which also work for a manifold stratified pair (X, B) with $\dim X \geq 5$ such that B has a mapping cylinder neighborhood in X. It seems likely that his techniques extend to arbitrary (high dimensional) manifold stratified pairs. At any rate, his work is further evidence for a moduli space interpretation of the results of this section.

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